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# A CONSTITUTIVE FRAME OF ELASTOPLASTICITY AT LARGE STRAINS BASED ON THE NOTION OF A PLASTIC METRIC†

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**Abstract**—The article presents a constitutive framework of large-strain elastoplasticity in both the Lagrangian and the Eulerian geometric setting which takes into account anisotropic material response. In summary, the key ingredients of this framework are: (i) the introduction of a plastic metric which is assumed to describe locally the history-dependent inelastic material response in the sense of an internal variable formulation. (ii) The definition of a convex elastic domain in the space of the local stress-like variable conjugate to the plastic metric, denoted as the plastic force. (iii) An equivalent Lagrangian and Eulerian representation of all constitutive functions as isotropic tensor functions in terms of an extended set of arguments, denoted as anisotropy variables. (iv) The set-up of normality rules for the evolution of the plastic metric and the anisotropy variables, yielding a canonical symmetric form of the elastoplastic tangent moduli. (v) A geometrically exact decomposition of the set of constitutive equations into possibly decoupled volumetric and isochoric contributions. Applications of the constitutive framework are demonstrated by means of several conceptual model problems which cover isotropic, initial anisotropic and induced anisotropic elastoplastic response. © 1998 Elsevier Science Ltd. All rights reserved.

## 1. INTRODUCTION

Foundations general frameworks and formalisms for the construction of constitutive equations for the phenomenological description of inelastic material response in solids have been proposed by Noll (1958, 1972), Truesdell and Toupin (1960), Truesdell and Noll (1965), Coleman and Gurtin (1967), Lubliner (1972, 1973), Coleman and Owen (1974), Krawietz (1986), among others, see also the references therein. Several sets of constitutive equations are established in the literature which cover the particular class of large-strain elastoplastic materials. We refer to the classical work of Green and Naghdi (1965, 1966), Lee (1969) and Mandel (1972, 1973) in this field, see also the review articles Hill (1978), Havner (1982), Asaro (1983) and Naghdi (1990) and the references cited therein. The textbooks Krawietz (1986), Lubliner (1990), Maugin (1992) and Havner (1992) give comprehensive and comparative introductions to the subject. In this context, the microstructural based theory for the description of finite elastoplastic deformations in ideal single crystals, often denoted as the continuum slip theory, seems to have achieved a stage of common acceptance in recent years, see for instance the work of Kröner (1960), Teodosiu (1970), Rice (1971), Mandel (1972), Teodosiu and Sidoroff (1976), Hill and Havner (1982), Havner (1982, 1992) and Asaro (1983). On the other hand, a broad variety of purely macroscopic theories of finite plasticity are controversially discussed in the literature. As pointed out by Naghdi (1990), there is strong disagreement between several existing schools on nearly all the relevant ingredients which form a finite plasticity theory. Topics under discussion are for instance invariance requirements, possible geometric settings, identification of plastic strains and elastic strains, possible decompositions of strain rates, the role of a plastic rotation and a plastic spin, possible formulations of yield criteria, stress space or strain space formulations, formulations of elastic response functions, yield criteria functions, flow rules and hardening laws, etc. Differences of possible approaches thus become particularly

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evident in the description of anisotropic elastoplastic material response. See for example Casey and Naghdi (1980, 1988), Lee (1980), Nemat-Nasser (1982), Mandel (1973), Dashner (1986) and Dafalias (1985) for a discussion of some of these issues. In addition, sound mathematically based theories for so-called materials with elastic range have been developed in recent years based on the ideas of Pipkin and Rivlin (1965), see Owen (1968, 1970), Šilhavý (1977), Lucchesi and Podio-Guidugli (1988, 1990) and Lucchesi *et al.* (1992). Geometrically exact algorithmic counterparts of nonlinear plasticity models suitable for the numerical simulation of initial boundary problems in context with application of finite element methods have been developed in the last decade. Due to the controversial approaches mentioned above, these algorithmic settings are often restricted to isotropic response where the different theoretical approaches can often be reconciled, as pointed out for example by Lubliner (1990) and Ibrahimbegović (1994). We refer in this context to the work of Simó and Ortiz (1985), Simó (1988, 1992), Simó and Miehe (1992), Weber and Anand (1990), Perić *et al.* (1990). Moran *et al.* (1990), Cuitiño and Ortiz (1992a, b), Miehe (1995a, 1996a, b) and the references cited therein.

The goal of this work is to present a constitutive framework of large-strain macroscopic elastoplasticity, suitable for large-scale numerical implementation, which takes into account anisotropic material response. It can be considered as a subclass of the very general theory of Green and Naghdi (1965) or the recent work of Antman (1995). However, it has some new methodical features which are, in the opinion of the author, very attractive with regard to the construction of concrete model problems, their numerical analysis and algorithmic implementation. The key ingredients of the framework proposed here are summarized in the abstract and discussed in the five items below.

(i) The notion of a plastic metric. In the first stage we introduce the notion of a plastic metric which is assumed to describe locally the history-dependence of the elastoplastic deformation process in the sense of an internal variable formulation. This notion turns out to be very helpful with regard to a geometric understanding of the theory and is particularly instructive for a setting up of concrete model problems. The evolution of the metric is governed by a constitutive evolution equation, denoted as the flow rule. As a typical initial condition we identify the plastic metric at the beginning of an elastoplastic deformation process with the natural metric of the material reference configuration. Note that the choice of a plastic metric as a symmetric, positive definite tensorial variable with six independent components excludes *a priori* a possible plastic rotation of local material elements. Thus, the frame presented here meets the invariance requirements discussed in Green and Naghdi (1971), Casey and Naghdi (1980) and Haupt (1985). Observe furthermore, that the constitutive framework discussed here avoids the notion of a "plastic intermediate configuration" as used by Lee (1969), Mandel (1972), Kratochvíl (1973) and others.

(ii) Elastic domain in the space of the plastic force. The next crucial step is the definition of a plastic force as the stress-like variable work-conjugate to the plastic metric. It appears in the part of the dissipation inequality which characterizes the local plastic power. Then, motivated by the structures of irreversible thermodynamics, a natural assumption is that this variable drives the evolution of the plastic metric. Consequently, we assume an elastic domain in the space of the plastic force governed by the traditional notion of a yield criterion function. This assumption is a key ingredient of a constitutive frame of finite plasticity presented here. In contrast to so-called strain-space formulations as proposed by Naghdi and Trapp (1975), Besdo (1981), Simó (1988), Antman (1995) and others, we adhere here to a traditional stress-space setting, which allows an adaptation of constitutive structures of the geometrically linear theory. The connection of the plastic force with the stresses depends on the choice of an elastic strain. Note that the elastic strain, which is assumed to govern the stored free energy of a local material element, constitutes a particular relationship between the natural metric on the current configuration and the plastic metric. The introduction of this type of relationship is unnecessary for a general set-up of the theoretical framework. We consider therefore, the definition of an elastic strain measure and the resulting identification of the plastic force in terms of the stress as an application of the theory. Examples are investigated in subsequent sections which are concerned with applications.

(iii) Representation in terms of isotropic tensor functions. A further key ingredient of the constitutive structure discussed here is its obvious invariance with respect to the geometric representation. We demonstrate this important property by considering throughout the whole paper in parallel the Lagrangian setting relative to the reference configuration and the Eulerian geometric setting relative to the current configuration. Note that the Eulerian setting can be attractive with regard to a numerical implementation for the following reason. If one chooses Cartesian coordinate charts, the current metric has a diagonal form in the Eulerian setting but is fully populated in the Lagrangian setting. The evaluation of Eulerian algorithmic representations of constitutive equations then needs less computational effort than the evaluation of their Lagrangian counterparts, see Simó and Miehe (1992) and Simó (1992). The essential step towards a geometrically unified representation of elastoplasticity is to force the constitutive functions to have the identical structure within the Lagrangian and Eulerian format. The only difference is the variables which enter the functions. These dual variables are related via fundamental geometric transformations between the Lagrangian and Eulerian configuration manifold and the associated local tangent and co-tangent spaces. An outcome of this approach is that the representations of the constitutive functions are strongly restricted due to the principle of material frame invariance which permits only representations as isotropic tensor functions. As a consequence, effects of anisotropic material response must be described by additional variables, denoted here as anisotropy variables. This implies an advantageous coordinate-free representation of anisotropy properties independent of particular choices of basis systems. In the case of initial anisotropic elastoplastic response, the anisotropy variables are simply defined as geometrical structural tensors, or material parameters with tensor character, with respect to the reference configuration. We refer in this context to approaches outlined e.g. in Doyle and Ericksen (1956) and Boehler (1987), see also the references therein. Note, in this context the general representation theory of isotropic functions as discussed, for example, in Wang (1969, 1970), Spencer (1971), Smith (1971), Betten (1982), Boehler (1987) and references therein. In the case of induced anisotropy, the variables develop during the elastoplastic deformation process and are governed by additional constitutive evolution equations.

(iv) Canonical symmetric constitutive structure. The elastic domain is assumed to be convex with respect to the plastic force and the anisotropy variables. We exploit this property by constructing a set of canonical evolution equations for the plastic metric and the anisotropy variables, denoted as normality rules. Following the conceptual work of Hill (1950), Drucker (1951), Ziegler (1963), Ziegler and Wehrli (1987) and Krawietz (1981), we formally construct these normality rules and the associated conditions for elastoplastic loading and unloading on the basis of a thermodynamic extremum principle. This principle, often denoted as the principle of maximum dissipation, is an equivalent to the normality of the evolution direction with respect to the yield surface and the convexity of the elastic domain. The exploitation of the principle uses standard tools of convex analysis. A particularly attractive feature is the simple extension of the rate-independent formulation to the particular class of rate-dependent elastoplastic materials considered by Perzyna (1971). This constitutive structure can formally be obtained by an approximation solution of thermodynamic extremum principle of the rate-independent theory. The most important property of a canonical set of constitutive equations with normality rules is the symmetry of the elastoplastic moduli. These moduli connect the rate of stress with the rate of total deformation. Their algorithmic counterparts play a key rule in iterative solvers for the numerical solution of initial boundary value problems of elastoplasticity. Thus, from the viewpoint of numerical analysis, the canonical constitutive structure is convenient because it needs less computational effort than a non-symmetric one. Clearly, it can only be applied if it fits the phenomenological response of the real material under consideration. The structure of the moduli within the large-strain formulation is not trivial. To the knowledge of the author, it is pointed out here for the first time in association with the key assumption of a flow criterion function formulated in terms of the plastic force.

(v) Decoupled volumetric–isochoric constitutive functions. A further key feature of the framework presented here is a geometrically exact decomposition of the constitutive functions into separate volumetric and isochoric contributions. This decomposition is motivated by the observation of real material response which is often completely different for the volumetric and isochoric deformation modes. We set up here a geometrically consistent approach to the decomposition of the constitutive functions based on the notion of a volumetric intermediate configuration. This notion induces modified objective rates associated with the isochoric part of the deformation only based on modified geometric transformations between the Lagrangian and the Eulerian manifold. To the knowledge of the author, these geometric relationships have not been outlined in the literature in the general terms presented here. It results in a split of the elastoplastic constitutive equations into a volumetric box governed by scalar functions and an isochoric box governed by tensor functions which we represent again in a Lagrangian as well as an Eulerian geometric setting. The approach offers in particular the possibility of a restriction of anisotropy properties to the isochoric part of the deformation.

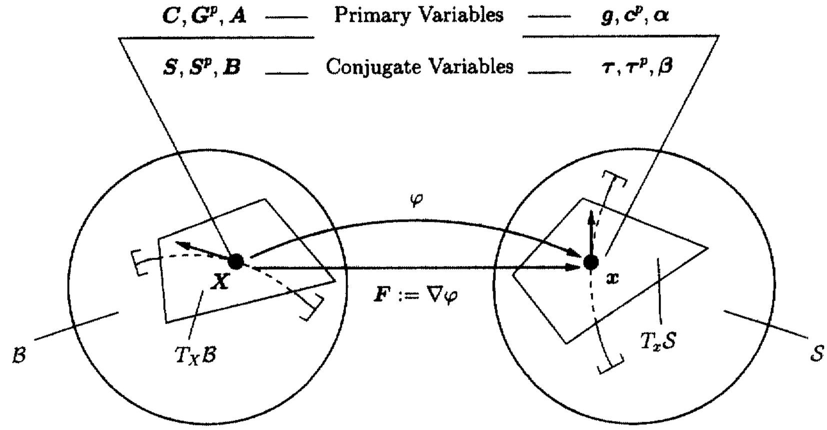
The paper is organised as follows. In Section 2 we set up the general theory within the canonical format mentioned above. This framework is then specified in Section 3 with regard to the decoupled volumetric–isochoric representation. Section 4 discusses possible definitions of elastic strain measures and the associated identifications of the plastic forces as a function of the stresses. Section 5 is devoted to an application of the theory to isotropic elastoplastic response. In this context, we suggest alternative representations in terms of invariants or eigenvalues of a mixed-variant elastic strain tensor. A key result within this context is a spectral representation of the isotropic elastoplastic constitutive equations in terms of dual covariant and contravariant eigenvector triads. Section 6 then discusses the modelling of initial elastic and plastic anisotropy within the framework proposed here. Here we consider anisotropic elastoplastic response with respect to the total deformation and, as mentioned above, a particular class of anisotropic elastoplastic response which is restricted to the isochoric part of the deformation. The modelling of initial anisotropy is demonstrated for the model problem of transverse isotropy. Finally, we demonstrate in Section 7 the application of the theory to the modelling of induced anisotropy phenomena. Here we consider as a model problem the Bauschinger effect or so-called kinematic hardening phenomenon and proposed a generalization of the classical Melan–Prager approach to the large strain format.

## 2. A GENERAL FRAMEWORK OF LARGE-STRAIN ELASTOPLASTICITY

In this section we propose a constitutive framework of large-strain elastoplasticity in a consistent thermodynamic setting in Lagrangian as well as Eulerian representation. The output is a box which determines the local stress response as an initial value problem in the sense of an internal variable formulation. As already mentioned above, the key ingredient is the representation of all constitutive functions as isotropic tensor functions. They are formulated in terms of a plastic metric and anisotropy variable, which describe the history dependence and effects of initial and induced anisotropy, respectively. Within the subsequent setup of the constitutive framework, we first introduce all the thermodynamic variables which are needed in the further development. Then we construct a canonical set of the elastoplastic constitutive equations based on an exploitation of thermodynamic principles within both the Lagrangian as well as the Eulerian geometric setting. Finally, we prove the symmetric structure of the developed canonical framework and make some remarks concerning a possibly non-symmetric structure.

### 2.1. *Notation and introduction of thermodynamic variables*

We start with the introduction of some basic geometric notation for the description of the model of large-strain elastoplasticity under consideration. After having defined fundamental kinematic mappings, we introduce all primary strain-like and conjugate stress-like thermodynamic variables which are needed in the subsequent development. The subsection concludes with definitions of time rates for the Lagrangian as well as the Eulerian




| | | |
|---|---|---|
| *Primary Variables:* | $[\cdot]^\flat$ | — $(\cdot)^\flat$ |
| *Current Metric* | $C$ | — $g$ |
| *Plastic Metric* | $G^p$ | — $c^p$ |
| *Anisotropy Variable* | $A$ | — $\alpha$ |
| *Conjugate Variables:* | $[\cdot]^\sharp$ | — $(\cdot)^\sharp$ |
| *Stress* | $S$ | — $\tau$ |
| *Plastic Force* | $S^p$ | — $\tau^p$ |
| *Anisotropy Variable* | $B$ | — $\beta$ |

Fig. 1. Lagrangian and Eulerian variables. Primary thermodynamic variables and work-conjugate dual variables on the Lagrangian and Eulerian configuration. The Eulerian variable $(\cdot)$ is connected with its dual Lagrangian counterpart $[\cdot]$ by a composition with the deformation map $\varphi$ and the associated linear tangent map $\mathbf{F}$ and normal map $\mathbf{F}^{-T}$. That is $(\cdot)^\flat = \mathbf{F}^{-T}[\cdot]^\flat\mathbf{F}^{-1}\circ\varphi^{-1}$ for covariant fields and $(\cdot)^\sharp = \mathbf{F}[\cdot]^\sharp\mathbf{F}^T\circ\varphi^{-1}$ for contravariant fields. Objective rates of Eulerian variables are defined by $\mathscr{L}_v(\cdot)^\flat := \mathbf{F}^{-T}\partial_t[\cdot]^\flat\mathbf{F}^{-1}\circ\varphi^{-1}$ and $\mathscr{L}_v(\cdot)^\sharp := \mathbf{F}\partial_t[\cdot]^\sharp\mathbf{F}^T\circ\varphi^{-1}$, respectively. The given reference metric is denoted with $\mathbf{G}$ and has the Eulerian form $\mathbf{c} = \mathbf{F}^{-T}\mathbf{G}\mathbf{F}^{-1}\circ\varphi^{-1}$.

geometric description. Figure 1 summarizes the notation and geometrical relationships introduce here.

2.1.1. *Fundamental kinematic mappings.* We consider elastoplastic deformations at large strains. Let $\mathscr{B} \subset \mathbb{R}^3$ be the reference configuration of the body of interest and

$$\varphi : \mathscr{B} \to \mathscr{S} \tag{1}$$

the nonlinear deformation map at time $t \in \mathbb{R}_+$. $\varphi$ maps points $X \in \mathscr{B}$ of the reference configuration $\mathscr{B}$ onto points $x = \varphi(X; t) \in \mathscr{S}$ of the current configuration $\mathscr{S} \subset \mathbb{R}^3$ as visualized in Fig. 1. The deformation gradient $\mathbf{F}(X; t) := \nabla_X\varphi(X; t)$ with Jacobian $J(X; t) := \det[\mathbf{F}(X; t)] > 0$ maps tangent vectors $\mathbf{T} \in T_X\mathscr{B}$ of material curves at $X$ onto tangent vectors $\mathbf{t} = \mathbf{F}\mathbf{T} \in T_x\mathscr{S}$ of the deformed material curves at $x$. Then $\mathbf{F}^{-T}(X; t)$ maps normal vectors $\mathbf{N} \in T^*_X\mathscr{B}$ at $X$ on material surfaces onto normal vectors $\mathbf{n} = \mathbf{F}^{-T}\mathbf{N} \in T^*_x\mathscr{S}$ at $x$ on the deformed material surfaces. Consequently, we denote $\mathbf{F}$ and $\mathbf{F}^{-T}$ as the tangent map and the normal map, respectively, and write

$$\mathbf{F} : T_X\mathscr{B} \to T_x\mathscr{S} \quad \text{and} \quad \mathbf{F}^{-T} : T^*_X\mathscr{B} \to T^*_x\mathscr{S}. \tag{2}$$

We consider $\mathscr{B}$ and $\mathscr{S}$ as differentiable manifolds with local tangent spaces $T_X\mathscr{B}$, $T_x\mathscr{S}$ and co-tangent spaces $T^*_X\mathscr{B}$, $T^*_x\mathscr{S}$ at $X \in \mathscr{B}$ and $x \in \mathscr{S}$, respectively. $\mathscr{B}$ and $\mathscr{S}$ are parameterized in terms of coordinate charts $\{X^A\}$ and $\{x^a\}$ within a neighbourhood $\mathscr{U}_X \subset \mathscr{B}$ and $\mathscr{U}_x \subset \mathscr{S}$ of the points $X$ and $x$, respectively, We refer to Doyle and Ericksen (1956), Marsden and Hughes (1983), Simó and Ortiz (1985) and Le and Stumpf (1993) for an addition reading concerning kinematic representation in terms of general coordinates. In the subsequent development we often use for $\mathscr{B}$ synonyms material configuration and Lagrangian manifold and for $\mathscr{S}$ spatial configuration or Eulerian manifold.

2.1.2. *Introduction of covariant metric tensors.* Now let $\mathbf{g}(x)$ and $\mathbf{G}(X)$ denote given covariant metric tensors on the current configuration and the reference configuration at $x$ and $X$, respectively. In what follows, we denote $\mathbf{g}$ as the current metric and $\mathbf{G}$ as the reference metric. The corresponding geometric objects on the dual configurations are obtained by a composition with the deformation map (1) and the associated normal map $(2)_2$. We denote the symmetric and positive definite tensors $\mathbf{C}(X;t)$ and $\mathbf{c}(x;t)$, defined by

$$\mathbf{C} := \mathbf{F}^T(\mathbf{g} \circ \varphi)\mathbf{F} \quad \text{and} \quad \mathbf{c} := \mathbf{F}^{-T}\mathbf{G}\mathbf{F}^{-1} \circ \varphi^{-1} \tag{3}$$

on the reference configuration and on the current configuration as convected current metric and as convected reference metric, respectively. The index representation associated with (3) relative to the coordinate charts $\{X^A\}$ and $\{x^a\}$ is $C_{AB} = (g_{ab} \circ \varphi)F^a{}_A F^b{}_B$ and $c_{ab} = G_{AB}F^{-1A}{}_a F^{-1B}{}_b \circ \varphi^{-1}$. In the literature, $\mathbf{C}$ is often denoted as the right Cauchy–Green tensor and $\mathbf{c}$ as the left Cauchy–Green tensor or inverse Finger tensor. Note that the current metric is the primary thermodynamic variable which governs the local stress power and therefore the elastic stress response. In the subsequent development we describe the history dependence of the inelastic material response by means of additional local variables. The evolution of these additional variables is then governed by constitutive evolution equations. Let $\mathbf{G}^p(X;t)$ with the initial condition $\mathbf{G}^p(X;t_0) = \mathbf{G}(X)$ at the initial time $t_0$ denote the covariant plastic metric on the Lagrangian manifold. Then the tensor field $\mathbf{c}^p(x;t)$ on the Eulerian manifold, defined by the composition

$$\mathbf{c}^p := \mathbf{F}^{-T}\mathbf{G}^p\mathbf{F}^{-1} \circ \varphi^{-1} \quad \text{with} \quad \mathbf{c}^p(x;t_0) = \mathbf{c}(x;t_0) \tag{4}$$

with the index form $c^p_{ab} = G^p_{AB}F^{-1A}{}_a F^{-1B}{}_b \circ \varphi^{-1}$, is called the convected plastic metric. $\mathbf{G}^p$ and $\mathbf{c}^p$ are assumed to be symmetric, positive definite tensor fields for the phenomenological description of the local plastic deformation. Note that this assumption restricts the model of finite elastoplasticity discussed here *a priori* to 6-dimensional flow rules. This is in contrast to the so-called intermediate configuration theories which include local plastic rotations. The evolution of the plastic metric $\mathbf{G}^p$ and $\mathbf{c}^p$ is determined by a plastic flow rule whose canonical structure will be discussed in the subsection below.

For the description of induced anisotropy effects, for example the Bauschinger phenomenon or non-isotropic damage accumulation, we consider as a model problem the generic covariant second-order tensorial anisotropy variable $\mathbf{A}(X;t)$ on the Lagrangian configuration with the possible initial condition $\mathbf{A}(X;t_0) = \mathbf{G}(X)$. $\mathbf{A}$ is assumed to be symmetric and positive definite. Then the Eulerian tensor field $\boldsymbol{\alpha}(x;t)$, defined as

$$\boldsymbol{\alpha} := \mathbf{F}^{-T}\mathbf{A}\mathbf{F}^{-1} \circ \varphi^{-1} \quad \text{with} \quad \boldsymbol{\alpha}(x;t_0) = \mathbf{c}(x;t_0) \tag{5}$$

and $\alpha_{ab} = A_{AB}F^{-1A}{}_a F^{-1B}{}_b \circ \varphi^{-1}$, is the convected anisotropy variable. The evolution of the anisotropy variable $\mathbf{A}$ and $\boldsymbol{\alpha}$ is determined by a constitutive evolution equation whose canonical structure will be discussed below.

2.1.3. *Introduction of contravariant conjugate variables.* For the thermodynamic description of large strain elastoplasticity we now introduce the thermodynamic variables conjugate to the current metric $(3)_1$, the plastic metric (4) and the internal variable (5) introduced above. Let $\boldsymbol{\tau}(x;t)$ denote the contravariant Eulerian Kirchhoff stress tensor work-conjugate to the current metric $\mathbf{c}$ introduced in (3). Then the Lagrangian field $\mathbf{S}(X;t)$, defined by the composition

$$\mathbf{S} := \mathbf{F}^{-1}(\boldsymbol{\tau} \circ \varphi)\mathbf{F}^{-T} \tag{6}$$

with the deformation map (1) and the tangent map $(2)_1$, is the symmetric Lagrangian–Piola convected stress tensor on the Lagrangian manifold conjugate to the convected current metric $\mathbf{C}$ in $(3)_1$. The index representation associated with (6) is $S^{AB} = (\tau^{ab} \circ \varphi)F^{-1A}{}_a F^{-1B}{}_b$. Let the Lagrangian field $\mathbf{S}^p(X;t)$ be the thermodynamic stress-like variable conjugate to the plastic metric $\mathbf{G}^p$ introduction in (4), in what follows denoted as the plastic force. The Eulerian field $\boldsymbol{\tau}^p(x;t)$, defined by

$$\boldsymbol{\tau}^p := \mathbf{F}\mathbf{S}^p\mathbf{F}^T \circ \varphi^{-1}, \tag{7}$$

with the index form $\tau^{pab} = S^{pAB}F^a{}_A F^b{}_B \circ \varphi^{-1}$ is the convected plastic force work-conjugate to the convected Eulerian plastic metric $\mathbf{c}^p$ in (4). Finally, we introduce the anisotropy variable $\mathbf{B}(X;t)$ work-conjugate to the internal variable $\mathbf{A}$ on the Lagrangian configuration, denoted as the internal force. Then $\boldsymbol{\beta}(x;t)$, obtained by the composition

$$\boldsymbol{\beta} := \mathbf{F}\mathbf{B}\mathbf{F}^T \circ \varphi^{-1} \tag{8}$$

with index representation $\beta^{ab} = B^{AB}F^a{}_A F^b{}_B \circ \varphi^{-1}$ is the convected internal force work-conjugate to the convected internal variable $\boldsymbol{\alpha}$ introduced in (5) on the Eulerian manifold.

2.1.4. *Material and spatial rates.* Throughout this paper, we consider the covariant tensors introduced in (3)–(5) as maps

$$[\cdot]^b \begin{cases} \mathbf{C} : T_X\mathscr{B} \times T_X\mathscr{B} \to \mathbb{R}_+ \\ \mathbf{G}^p : T_X\mathscr{B} \times T_X\mathscr{B} \to \mathbb{R}_+ \\ \mathbf{A} : T_X\mathscr{B} \times T_X\mathscr{B} \to \mathbb{R}_+ \end{cases} \quad \text{and} \quad (\cdot)^b \begin{cases} \mathbf{g} : T_x\mathscr{S} \times T_x\mathscr{S} \to \mathbb{R}_+ \\ \mathbf{c}^p : T_x\mathscr{S} \times T_x\mathscr{S} \to \mathbb{R}_+ \\ \boldsymbol{\alpha} : T_x\mathscr{S} \times T_x\mathscr{S} \to \mathbb{R}_+ \end{cases}. \tag{9}$$

The contravariant conjugate thermodynamic variables introduced in (6)–(8) are viewed as maps

$$[\cdot]^\# \begin{cases} \mathbf{S} : T^*_X\mathscr{B} \times T^*_X\mathscr{B} \to \mathbb{R} \\ \mathbf{S}^p : T^*_X\mathscr{B} \times T^*_X\mathscr{B} \to \mathbb{R} \\ \mathbf{B} : T^*_X\mathscr{B} \times T^*_X\mathscr{B} \to \mathbb{R} \end{cases} \quad \text{and} \quad (\cdot)^\# \begin{cases} \boldsymbol{\tau} : T^*_x\mathscr{S} \times T^*_x\mathscr{S} \to \mathbb{R} \\ \boldsymbol{\tau}^p : T^*_x\mathscr{S} \times T^*_x\mathscr{S} \to \mathbb{R} \\ \boldsymbol{\beta} : T^*_x\mathscr{S} \times T^*_x\mathscr{S} \to \mathbb{R} \end{cases}. \tag{10}$$

Now let the symbols $[\cdot]$ and $(\cdot)$ introduced in the equations above denote dual Lagrangian and Eulerian tensorial variables defined on $\mathscr{B}$ and $\mathscr{S}$, respectively. They are connected through the geometric transformations (3)–(8) governed by the nonlinear deformation map (1) and the linear tangent and normal maps (2). Following standard terminologies of differential geometry, we denote these transformations of dual geometric objects on the Lagrangian and Eulerian manifold as pull back and push forward transformations, using the symbolic notation $[\cdot] = \varphi^*(\cdot)$ and $(\cdot) = \varphi_*[\cdot]$. Now let us denote with $\partial_t[\cdot]$ the time rates of Lagrangian objects and with $\mathscr{L}_v(\cdot)$ objective rates of the associated Eulerian objects. We choose as objective rates the Lie derivatives of the Eulerian objects, which are obtained from the time derivatives of the associated Lagrangian objects by a push-forward transformation, i.e. $\mathscr{L}_v(\cdot) := \varphi_*\{\partial_t[\varphi^*(\cdot)]\} = \varphi_*\{\partial_t[\cdot]\}$. For second-order tensors, we obtain the relationships

$$\begin{aligned} \mathscr{L}_v(\cdot)^b &= \dot{\overline{(\cdot)}}{}^b + \mathbf{l}^T(\cdot)^b + (\cdot)^b\mathbf{l} \\ \mathscr{L}_v(\cdot)^\# &= \dot{\overline{(\cdot)}}{}^\# - \mathbf{l}(\cdot)^\# - (\cdot)^\#\mathbf{l}^T \end{aligned} \tag{11}$$

for covariant fields and contravariant fields, respectively, see also Fig. 1. Here $\dot{(\cdot)}$ denotes the material or total time derivative of the Eulerian field $(\cdot)$ and

$$\mathbf{l} := \partial_t \mathbf{F}\mathbf{F}^{-1} \circ \varphi^{-1} \tag{12}$$

is the spatial velocity gradient. For the second-order tensor under consideration, the Lie derivative defined in (11) are denoted in the literature as Oldroyd rates. Finally observe the particular forms of the Lie derivatives of the convected reference metric $\mathscr{L}_v \mathbf{c} \equiv 0$ due to $\partial_t \mathbf{G} \equiv 0$ and the current metric $\mathscr{L}_v \mathbf{g} = \mathbf{l}^T \mathbf{g} + \mathbf{g}\mathbf{l} = 2\,\text{sym}\,[\mathbf{g}\mathbf{l}]$ due to $\dot{\mathbf{g}} \equiv 0$.

### 2.2. *A canonical set of constitutive equations*

We now develop a set of constitutive equations in terms of the variables introduced above. Thereby, we first focus on the partial elastic response governed by a stored free energy function. Then canonical evolution equations for the plastic metric and the anisotropy variable are constructed by exploitation of a plastic potential function, taking into account a thermodynamical extremum principle.

2.2.1. *Local elastic constitutive response functions.* Let $\psi$ denote the change in locally stored free energy during the deformation process from the reference configuration to the current configuration. We assume a functional dependence on the current metric, the anisotropy variable, the plastic metric and the reference metric introduced in $(3)_1$, (5), (4) and $(3)_2$, respectively. Here, we consider the particular form

$$\boxed{\psi = \hat{\psi}(\mathbf{C}, \mathbf{A}\,; \mathbf{G}^p, G, X) = \hat{\psi}(\mathbf{g}, \boldsymbol{\alpha}\,; \mathbf{c}^p, \mathbf{c}, x).} \tag{13}$$

Thus, we assume the identical function $\hat{\psi}$ in the Lagrangian and Eulerian geometric setting, resulting in an identical structure of the Lagrangian and Eulerian set of constitutive equations as pointed out in the subsequent development. The function (13) is then strongly restricted due to the principal of material frame invariance. It demands in its active form invariance with respect to rigid body motions superimposed onto the current configuration. Therefore, we consider the modifications $\mathbf{F}^+ := \mathbf{Q}\mathbf{F}$ and $\mathbf{F}^{-T+} := \mathbf{Q}\mathbf{F}^{-T}$ of the local tangent and normal map (2) where $\mathbf{Q} \in SO_3$ is an arbitrary rotation, i.e. an element of the special orthogonal group $SO_3 := \{\mathbf{Q} \mid \mathbf{Q}^T\mathbf{Q} = \mathbf{1} \text{ and } \det[\mathbf{Q}] = 1\}$. Under this rotation, the Eulerian variables $\mathbf{g}$, $\boldsymbol{\alpha}$, $\mathbf{c}^p$ and $\mathbf{c}$ transform as $\mathbf{g}^+ := \mathbf{Q}\mathbf{g}\mathbf{Q}^T$, $\boldsymbol{\alpha}^+ := \mathbf{Q}\boldsymbol{\alpha}\mathbf{Q}^T$, $\mathbf{c}^{p+} := \mathbf{Q}\mathbf{c}^p\mathbf{Q}^T$ and $\mathbf{c}^+ := \mathbf{Q}\mathbf{c}\mathbf{Q}^T$. This is a consequence of the definitions $(3)_1$, (5), (4) and $(3)_2$. Thus, the principle of material frame invariance demands

$$\hat{\psi}(\mathbf{Q}\mathbf{g}\mathbf{Q}^T, \mathbf{Q}\boldsymbol{\alpha}\mathbf{Q}^T; \mathbf{Q}\mathbf{c}^p\mathbf{Q}^T, \mathbf{Q}\mathbf{c}\mathbf{Q}^T, x) = \hat{\psi}(\mathbf{g}, \boldsymbol{\alpha}\,; \mathbf{c}^p, \mathbf{c}, x) \quad \forall \mathbf{Q} \in SO_3 \tag{14}$$

$SO_3$-invariance of all tensorial slots of the free energy function $\hat{\psi}$. Recalling the assumption (13), we conclude in particular the necessity of $SO_3$-invariance of the free energy function with respect to the Lagrangian variables

$$\hat{\psi}(\mathbf{Q}\mathbf{C}\mathbf{Q}^T, \mathbf{Q}\mathbf{A}\mathbf{Q}^T; \mathbf{Q}\mathbf{G}^p\mathbf{Q}^T, \mathbf{Q}\mathbf{G}\mathbf{Q}^T, X) = \hat{\psi}(\mathbf{C}, \mathbf{A}\,; \mathbf{G}^p, \mathbf{G}, X) \quad \forall \mathbf{Q} \in SO_3. \tag{15}$$

Thus, the principle of material frame invariance restricts the free energy function of the structure (13) to an isotropic tensor function of its tensorial arguments. According to the representation theorems of isotropic tensor functions, see for instance Boehler (1987) and references therein, the function (13) can depend only on the coupled invariants of its tensorial arguments which form an irreducible functional basis. Thus, ansatz (13) has a crucial consequence for the conceptual treatment of elastic anisotropy effects. Anisotropic elastic response is described based on isotropic tensor functions with an extended set of arguments, referred to in what follows as anisotropy tensors or structural tensors. A generic anisotropy tensor is the second-order variable $\mathbf{A}$ with the Eulerian counterpart $\boldsymbol{\alpha}$ introduced in (5) which offers for instance the description of initial transversely anisotropic elastic response or orientated induced elastic anisotropy phenomena due to damage evolution. More complicated initial anisotropic stress response of orientated damage needs possibly the introduction of more anisotropy tensors, possibly the introduction of fourth or even higher order anisotropy tensors. However, the conceptual approach does not differ from that outlined in the subsequent development in terms of a single second-order tensorial variable **A**. Particular representations and restrictions of the free energy function are discussed in Sections 5–7.

The evolution of the free energy takes, based on assumption (13), the Lagrangian and Eulerian form

$$\begin{aligned}\partial_t\psi &= \partial_{\mathbf{C}}\hat{\psi}:\partial_t\mathbf{C}+\partial_{\mathbf{G}^p}\hat{\psi}:\partial_t\mathbf{G}^p+\partial_{\mathbf{A}}\hat{\psi}:\partial_t\mathbf{A}\\ &= \partial_{\mathbf{g}}\hat{\psi}:\mathscr{L}_v\mathbf{g}+\partial_{\mathbf{c}^p}\hat{\psi}:\mathscr{L}_v\mathbf{c}^p+\partial_{\boldsymbol{\alpha}}\hat{\psi}:\mathscr{L}_v\boldsymbol{\alpha}\end{aligned} \tag{16}$$

in terms of the rates of the Lagrangian and Eulerian variables defined in Section 2.1.4 above. Thereby, the Eulerian rate form is obtained from the Lagrangian rate form by operations of the type $\partial_{\mathbf{C}}\hat{\psi}:\partial_t\mathbf{C} = [\mathbf{F}\partial_{(\mathbf{F}^T\mathbf{gF})}\hat{\psi}\mathbf{F}^T]:[\mathbf{F}^{-T}\partial_t\mathbf{C}\mathbf{F}^{-1}] = \partial_{\mathbf{g}}\hat{\psi}:\mathscr{L}_v\mathbf{g}$. We construct the set of constitutive equations in a way that it satisfies *a priori* the second axiom of thermodynamics which postulates a positive entropy production. We use here as a local form of the second axiom the so-called Clausius–Planck inequality for the internal dissipation, see e.g. Truesdell and Noll (1965) p. 295, which degenerates in the isothermal case under consideration to the form

$$\mathscr{D}^p := \mathbf{S}:\tfrac{1}{2}\partial_t\mathbf{C}-\partial_t\psi = \boldsymbol{\tau}:\tfrac{1}{2}\mathscr{L}_v\mathbf{g}-\partial_t\psi \geqslant 0 \tag{17}$$

in the Lagrangian and the Eulerian geometric setting, respectively. **S** and $\boldsymbol{\tau}$ are the stress fields introduced in (6). The well-known inequality says that the local stress power $\mathbf{S}:\frac{1}{2}\partial_t\mathbf{C} = \boldsymbol{\tau}:\frac{1}{2}\mathscr{L}_v\mathbf{g}$ is greater than or equal to the evolution $\partial_t\psi$ of the local energy storage, where the equality defines hyperelastic material response. The insertion of the evolution (16) into (17) yields

$$\begin{aligned}\mathscr{D}^p &= [\mathbf{S}-2\partial_{\mathbf{C}}\hat{\psi}]:\tfrac{1}{2}\partial_t\mathbf{C}-\partial_{\mathbf{G}^p}\hat{\psi}:\partial_t\mathbf{G}^p-\partial_{\mathbf{A}}\hat{\psi}:\partial_t\mathbf{A}\\ &= [\boldsymbol{\tau}-2\partial_{\mathbf{g}}\hat{\psi}]:\tfrac{1}{2}\mathscr{L}_v\mathbf{g}-\partial_{\mathbf{c}^p}\hat{\psi}:\mathscr{L}_v\mathbf{c}^p-\partial_{\boldsymbol{\alpha}}\hat{\psi}:\mathscr{L}_v\boldsymbol{\alpha} \geqslant 0.\end{aligned} \tag{18}$$

A standard argumentation, following Coleman and Gurtin (1967) and Lubliner (1990), is as follows. A local elastoplastic process can be, as a special case, purely elastic in an arbitrary time interval of the deformation history. Within this interval, the evolution of the plastic metric and the evolution of the anisotropy variable is assumed to be zero, i.e. $\partial_t\mathbf{G}^p = \partial_t\mathbf{A} = \mathscr{L}_v\mathbf{c}^p = \mathscr{L}_v\boldsymbol{\alpha} = 0$. Furthermore, the inequality (18) degenerates then to an equality which has to be satisfied for arbitrary rates $\frac{1}{2}\partial_t\mathbf{C}$ and $\frac{1}{2}\mathscr{L}_v\mathbf{g}$ of the local deformation. As a consequence of these assumptions, the brackets in (18) must vanish for all times, yielding the local hyperelastic constitutive functions for the stresses **S** and $\boldsymbol{\tau}$ in Table 1.

Table 1. A canonical constitutive set of anisotropic elastoplasticity

| | Lagrangian setting | Eulerian setting |
|---|---|---|
| *Free energy* | $\psi = \hat{\psi}(\mathbf{C},\mathbf{A};\mathbf{G}^p,\mathbf{G},X)$ | $\psi = \hat{\psi}(\mathbf{g},\boldsymbol{\alpha};\mathbf{c}^p,\mathbf{c},x)$ |
| Stresses | $\mathbf{S} = 2\partial_{\mathbf{C}}\hat{\psi}$ | $\boldsymbol{\tau} = 2\partial_{\mathbf{g}}\hat{\psi}$ |
| Plastic force | $\mathbf{S}^p := -\partial_{\mathbf{G}^p}\hat{\psi}$ | $\boldsymbol{\tau}^p := -\partial_{\mathbf{c}^p}\hat{\psi}$ |
| Internal force | $\mathbf{B} := -\partial_{\mathbf{A}}\hat{\psi}$ | $\boldsymbol{\beta} := -\partial_{\boldsymbol{\alpha}}\hat{\psi}$ |
| *Yield function* | $\phi = \hat{\phi}(\mathbf{S}^p,\mathbf{B};\mathbf{G}^p,\mathbf{G},X)$ | $\phi = \hat{\phi}(\boldsymbol{\tau}^p,\boldsymbol{\beta};\mathbf{c}^p,\mathbf{c},x)$ |
| Flow rule | $\partial_t\mathbf{G}^p = \lambda\partial_{\mathbf{S}^p}\hat{\phi}$ | $\mathscr{L}_v\mathbf{c}^p = \lambda\partial_{\boldsymbol{\tau}^p}\hat{\phi}$ |
| Evolution | $\partial_t\mathbf{A} = \lambda\partial_{\mathbf{B}}\hat{\phi}$ | $\mathscr{L}_v\boldsymbol{\alpha} = \lambda\partial_{\boldsymbol{\beta}}\hat{\phi}$ |
| Loading | $\lambda \geqslant 0;\ \hat{\phi} \leqslant 0;\ \lambda\hat{\phi} = 0$ | $\lambda \geqslant 0;\ \hat{\phi} \leqslant 0;\ \lambda\hat{\phi} = 0$ |
| [Viscoplastic | $\lambda := (1/\eta)\hat{\rho}'(\phi^+)$ | $\lambda := (1/\eta)\hat{\rho}'(\phi^+)$] |

Taking into account this result, the Clausius–Planck inequality (18) takes the reduced form

$$\hat{\mathscr{D}}^p = \mathbf{S}^p : \partial_t \mathbf{G}^p + \mathbf{B} : \partial_t \mathbf{A} = \boldsymbol{\tau}^p : \mathscr{L}_v \mathbf{c}^p + \boldsymbol{\beta} : \mathscr{L}_v \boldsymbol{\alpha} \geqslant 0 \tag{19}$$

where we have introduced per definition the variables $\mathbf{S}^p$ and $\boldsymbol{\tau}^p$, denoted as plastic forces, and $\mathbf{B}$ and $\boldsymbol{\beta}$, denoted as internal forces, with the identification in Table 1. In what follows, we refer to $\hat{\mathscr{D}}^p$ as the dissipation function. It is an inner product of thermodynamic forces and fluxes. The dissipation function plays a fundamental role in the subsequent development.

2.2.2. *Local plastic constitutive response functions.* What remains is the determination of the evolution equations for the internal metric and the internal variable. In what follows we derive the canonical form of these evolution equations which result in a symmetric form of the elastoplastic tangent moduli as proven in Section 2.3. These canonical evolution equations can be interpreted as normality rules with respect to a convex plastic potential function.

As a main characteristic of elastoplastic and viscoelastoplastic material response, we consider an elastic domain in the stress space. One of the key aspects of this work is the formulation of the domain in terms of the thermodynamic forces which drive the plastic dissipation in the reduced dissipation inequality (19). Thus, we assume an elastic domain

$$\mathbb{E}^L := \{(\mathbf{S}^p, \mathbf{B}) \in \mathbb{R}^6 \times \mathbb{R}^6 \mid \hat{\phi}(\mathbf{S}^p, \mathbf{B}; \mathbf{G}^p, \mathbf{G}, X) \leqslant 0\} \tag{20}$$

in the space of the Lagrangian thermodynamic forces $(\mathbf{S}^p, \mathbf{B})$ and alternatively

$$\mathbb{E}^E := \{(\boldsymbol{\tau}^p, \boldsymbol{\beta}) \in \mathbb{R}^6 \times \mathbb{R}^6 \mid \hat{\phi}(\boldsymbol{\tau}, \boldsymbol{\beta}; \mathbf{c}^p, \mathbf{c}, x) \leqslant 0\} \tag{21}$$

in the space of the Eulerian thermodynamic forces $(\boldsymbol{\tau}^p, \boldsymbol{\beta})$. Here $\hat{\phi}$ is a yield function, which is assumed to depend on the thermodynamic forces, the plastic metric and possibly the reference metric. In analogy to the constitutive ansatz (13) of the free energy we assume

$$\boxed{\phi = \hat{\phi}(\mathbf{S}^p, \mathbf{B}; \mathbf{G}^p, \mathbf{G}, X) = \hat{\phi}(\boldsymbol{\tau}^p, \boldsymbol{\beta}; \mathbf{c}^p, \mathbf{c}, x)} \tag{22}$$

the identical function in the Lagrangian and the Eulerian geometric setting, resulting in an identical structure of the evolution equations in the Lagrangian and Eulerian format. Then the yield function (22) is strongly restricted due to the principle of material frame invariance. Considering again an arbitrary rotation $\mathbf{Q} \in SO_3$ superimposed onto the current configuration, inducing the modifications $\mathbf{F}^+ := \mathbf{QF}$ and $\mathbf{F}^{-T} := \mathbf{QF}^{-T}$ of the local tangent and normal map (2), the Eulerian variables $\boldsymbol{\tau}^p$, $\boldsymbol{\beta}$, $\mathbf{c}^p$ and $\mathbf{c}$ transform as $\boldsymbol{\tau}^{p+} := \mathbf{Q}\boldsymbol{\tau}^p\mathbf{Q}^T$, $\boldsymbol{\beta}^+ := \mathbf{Q}\boldsymbol{\beta}\mathbf{Q}^T$, $\mathbf{c}^{p+} := \mathbf{Q}\mathbf{c}^p\mathbf{Q}^T$ and $\mathbf{c}^+ := \mathbf{QcQ}^T$. This is a consequence of the definitions (7), (8), (4) and $(3)_2$. Thus, the principal of material frame invariance demands

$$\hat{\phi}(\mathbf{Q}\boldsymbol{\tau}^p\mathbf{Q}^T, \mathbf{Q}\boldsymbol{\beta}\mathbf{Q}^T; \mathbf{Q}\mathbf{c}^p\mathbf{Q}^T, \mathbf{QcQ}^T, x) = \hat{\phi}(\boldsymbol{\tau}^p, \boldsymbol{\beta}; \mathbf{c}^p, \mathbf{c}, x) \quad \forall \mathbf{Q} \in SO_3. \tag{23}$$

Thus, $\hat{\phi}$ is an isotropic tensor function and the ansatz (22) induces in particular full $SO_3$-invariance of the free energy function with respect to the Lagrangian variables, i.e.

$$\hat{\phi}(\mathbf{Q}\mathbf{S}^p\mathbf{Q}^T, \mathbf{QBQ}^T; \mathbf{Q}\mathbf{G}^p\mathbf{Q}^T, \mathbf{QGQ}^T, X) = \hat{\phi}(\mathbf{S}^p, \mathbf{B}; \mathbf{G}^p, \mathbf{G}, X) \quad \forall \mathbf{Q} \in SO_3. \tag{24}$$

According to the representation theorems of isotropic tensor functions, the function (22) can depend only on the coupled invariants of its tensorial arguments which form an irreducible functional basis. Anisotropic plastic response is described based on isotropic tensor functions with an extended set of arguments, referred to in what follows as anisotropy tensors or structural tensors. A generic anisotropy tensor is the second-order variable $\mathbf{B}$

with the Eulerian counterpart $\boldsymbol{\beta}$ introduced in (8) which offers for instance the description of initial transversely plastic anisotropic response or induced anisotropy phenomena like Bauschinger's effect. More complicated initial anisotropic stress response of orientated damage needs the introduction of more anisotropy tensors, eventually the introduction of fourth or even higher order anisotropy tensors. However, the conceptual approach does not differ from that outlined in the subsequent development in terms of the second-order tensorial variable **B**. Particular representations and restrictions of the yield function are discussed in Sections 5–7.

In order to postulate canonical evolution equations, we consider the yield function (22) as a flow hypersurface in the space of the thermodynamic forces, evaluated with the plastic metric and the reference metric. The canonical evolution equations are then derived from the argument.

$$\begin{aligned}\hat{\mathscr{D}}^p - \mathscr{D}^{p*} &:= [\mathbf{S}^p - \mathbf{S}^{p*}] : \partial_t \mathbf{G}^p + [\mathbf{B} - \mathbf{B}^*] : \partial_t \mathbf{A} \\ &= [\boldsymbol{\tau}^p - \boldsymbol{\tau}^{p*}] : \mathscr{L}_v \mathbf{c}^p + [\boldsymbol{\beta} - \boldsymbol{\beta}^*] : \mathscr{L}_v \boldsymbol{\alpha} \geqslant 0 \end{aligned} \tag{25}$$

for all admissible variations $(\mathbf{S}^{p*}, \mathbf{B}^*) \in \mathbb{E}^L$ of the Lagrangian thermodynamic forces and $(\boldsymbol{\tau}^{p*}, \boldsymbol{\beta}^*) \in \mathbb{E}^E$ of the Eulerian thermodynamic forces. A geometric interpretation of this inequality is given in Fig. 2. Applied to the true stresses, this principle is known in plasticity theory as the so-called principle of maximum plastic dissipation, see for example Hill (1950), Drucker (1951), Mandel (1972), Lubliner (1990), Simó and Miehe (1992). We use it here in a more general context by application to the full vectors of thermodynamic forces. This is in line with conceptual frameworks outline in Ziegler (1963), Krawietz (1981) and Ziegler and Wehrli (1987). The principle is equivalent to the convexity of the yield function with respect to the thermodynamic forces and the normality of the evolution equations for the thermodynamic forces with respect to the flow hypersurface $\hat{\phi} = 0$ in the space of the thermodynamic forces.

The evolution equations can be formally derived from a saddle point problem based on the Lagrange function

$$\hat{\mathscr{L}} = -\hat{\mathscr{D}}^p + \lambda \hat{\phi} \rightarrow \text{stat.} \tag{26}$$

defined on the Lagrangian manifold at $X \in \mathscr{B}$ in terms of the Lagrangian variables as well as on the Eulerian manifold at $x \in \mathscr{S}$ in terms of the Eulerian variables, according to the representation of (19) and (22). This function transforms the constrained optimization problem (25) into an unconstrained saddle point problem. The solution to the problem

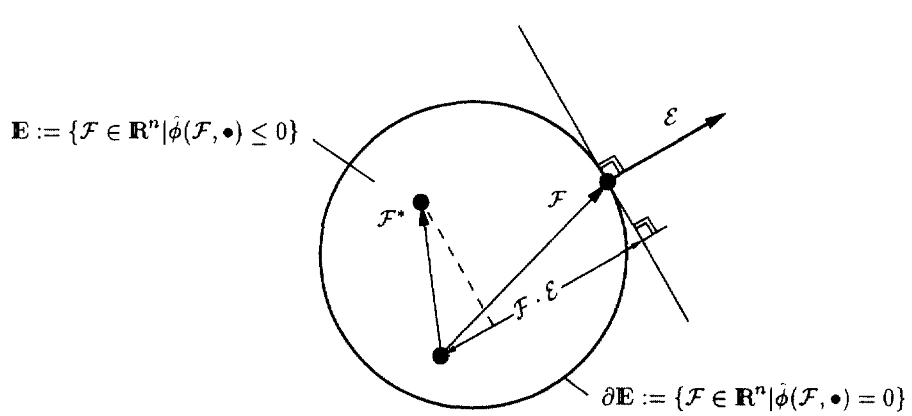



Fig. 2. Thermodynamic extremal principle. Let $\mathbb{E}$ be a convex elastic domain in the space of the thermodynamic forces $\mathscr{F}$, characterized by a hypersurface $\hat{\phi}(\mathscr{F}; \bullet) = 0$. The principle $\mathscr{F} \cdot \mathscr{E} \geqslant \mathscr{F}^* \cdot \mathscr{E} \forall \mathscr{F}^* \in \mathbb{E}$ forces the thermodynamic flux vector $\mathscr{E}$ to be normal to the hypersurface $\hat{\phi}(\mathscr{F}; \bullet) = 0$. Then for smooth functions $\hat{\phi}$ the flux $\mathscr{E}$ is proportional to the gradient $\partial_{\mathscr{F}} \hat{\phi}$.

(26) is given by the Kuhn–Tucker equations, see for example the standard literature of nonlinear convex analysis. The gradients

$$\left.\begin{array}{l}\partial_{\mathbf{S}^p}\hat{\mathscr{L}} = 0\\ \partial_{\mathbf{B}}\hat{\mathscr{L}} = 0\end{array}\right\} \quad \text{and} \quad \left.\begin{array}{l}\partial_{\tau^p}\hat{\mathscr{L}} = 0\\ \partial_{\beta}\hat{\mathscr{L}} = 0\end{array}\right\} \tag{27}$$

yield the canonical evolution equations for the plastic metric and the anisotropy variable in the Lagrangian and Eulerian setting, respectively, as outlined in Table 1. They are completed by the Kuhn–Tucker-type loading–unloading conditions

$$\lambda \geqslant 0; \quad \hat{\phi} \leqslant 0; \quad \lambda\hat{\phi} = 0 \tag{28}$$

which determine the plastic parameter $\lambda$.

The classical form of Perzyna-type viscoelastoplasticity, see e.g. Perzyna (1971), can be formally derived if the constrained optimization problem (25) is solved approximately based on a penalty approach. We therefore introduce as an alternative to the Lagrange function (26) the penalty functional

$$\hat{\mathscr{P}} = -\hat{\mathscr{D}}^p + \frac{1}{\eta}\hat{p}(\hat{\phi}^+) \rightarrow \text{stat.} \tag{29}$$

defined on the Lagrangian manifold at $X \in \mathscr{B}$ in terms of the Lagrangian variables as well as on the Eulerian manifold at $x \in \mathscr{S}$ in terms of the Eulerian variables, according to the representation of (19) and (22). The penalty parameter $(1/\eta) \in (0, \infty)$ is interpreted as a scalar material parameter denoted in what follows as the viscosity. The constitutive function $\hat{p}\colon \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a monotonic increasing $C^1$ penalty function which satisfies the condition $\hat{p}(0) = 0$. Furthermore, we denote

$$\hat{\phi}^+ := \begin{cases} 0 & \text{for } \hat{\phi} \leqslant 0, \\ \hat{\phi} & \text{for } \hat{\phi} > 0 \end{cases} \tag{30}$$

as the viscoplastic loading function. The gradients of the penalty function

$$\left.\begin{array}{l}\partial_{\mathbf{S}^p}\hat{\mathscr{P}} = 0\\ \partial_{\mathbf{B}}\hat{\mathscr{P}} = 0\end{array}\right\} \quad \text{and} \quad \left.\begin{array}{l}\partial_{\tau^p}\hat{\mathscr{P}} = 0\\ \partial_{\beta}\hat{\mathscr{P}} = 0\end{array}\right\} \tag{31}$$

yield the canonical evolution equations for the plastic metric and the internal variable in the Lagrangian and Eulerian setting, respectively, as pointed out in Table 1. The only difference to the case of plasticity is that the plastic parameter is now per definition determined by the constitutive expression

$$\lambda = \frac{1}{\eta}\hat{p}'[\phi^+] \tag{32}$$

in terms of the viscosity $1/\eta$ and the constitutive function $\hat{p}$ of the viscoplastic loading $\phi^+$.

### 2.3. *Elastoplastic tangent moduli*

We consider now rate equations for the Lagrangian and Eulerian stresses, respectively, in the form

$$\partial_t \mathbf{S} = \mathbb{C}^{ep} : \tfrac{1}{2}\partial_t \mathbf{C} \quad \text{and} \quad \mathscr{L}_v \boldsymbol{\tau} = \mathbb{c}^{ep} : \tfrac{1}{2}\mathscr{L}_v \mathbf{g}. \tag{33}$$

The goal of this subsection is the determination of the fourth-order tensors $\mathbb{C}^{ep}$ and $\mathbb{c}^{ep}$ in

(33), to which we refer in what follows as the Lagrangian and Eulerian elastoplastic tangent moduli or generalized Prandtl–Reuss tensors. They connect the rate of stress with the rate of total deformation. Therefore, consider first the rate equations of the stresses and thermodynamic forces introduced in Table 1

$$\left.\begin{aligned} \partial_t \mathbf{S} &= 4\partial^2_{\mathbf{CC}}\hat{\psi} : \tfrac{1}{2}\partial_t \mathbf{C} - \lambda[2\partial^2_{\mathbf{CG}^p}\hat{\psi} : \partial_{\mathbf{S}^p}\hat{\phi} + 2\partial^2_{\mathbf{CA}}\hat{\psi} : \partial_{\mathbf{B}}\hat{\phi}] \\ \partial_t \mathbf{S}^p &= 2\partial^2_{\mathbf{G}^p\mathbf{C}}\hat{\psi} : \tfrac{1}{2}\partial_t \mathbf{C} - \lambda[\partial^2_{\mathbf{G}^p\mathbf{G}^p}\hat{\psi} : \partial_{\mathbf{S}^p}\hat{\phi} + \partial^2_{\mathbf{G}^p\mathbf{A}}\hat{\psi} : \partial_{\mathbf{B}}\hat{\phi}] \\ \partial_t \mathbf{B} &= 2\partial^2_{\mathbf{AC}}\hat{\psi} : \tfrac{1}{2}\partial_t \mathbf{C} - \lambda[\partial^2_{\mathbf{AG}^p}\hat{\psi} : \partial_{\mathbf{S}^p}\hat{\phi} + \partial^2_{\mathbf{AA}}\hat{\psi} : \partial_{\mathbf{B}}\hat{\phi}] \end{aligned}\right\} \tag{34}$$

within the Lagrangian setting and

$$\left.\begin{aligned} \mathscr{L}_v \boldsymbol{\tau} &= 4\partial^2_{\mathbf{gg}}\hat{\psi} : \tfrac{1}{2}\mathscr{L}_v \mathbf{g} - \lambda[2\partial^2_{\mathbf{gc}^p}\hat{\psi} : \partial_{\boldsymbol{\tau}^p}\hat{\phi} + 2\partial^2_{\mathbf{g}\boldsymbol{\alpha}}\hat{\psi} : \partial_{\boldsymbol{\beta}}\hat{\phi}] \\ \mathscr{L}_v \boldsymbol{\tau}^p &= 2\partial^2_{\mathbf{c}^p\mathbf{g}}\hat{\psi} : \tfrac{1}{2}\mathscr{L}_v \mathbf{g} - \lambda[\partial^2_{\mathbf{c}^p\mathbf{c}^p}\hat{\psi} : \partial_{\boldsymbol{\tau}^p}\hat{\phi} + \partial^2_{\mathbf{c}^p\boldsymbol{\alpha}}\hat{\psi} : \partial_{\boldsymbol{\beta}}\hat{\phi}] \\ \mathscr{L}_v \boldsymbol{\beta} &= 2\partial^2_{\boldsymbol{\alpha}\mathbf{g}}\hat{\psi} : \tfrac{1}{2}\mathscr{L}_v \mathbf{g} - \lambda[\partial^2_{\boldsymbol{\alpha}\mathbf{c}^p}\hat{\psi} : \partial_{\boldsymbol{\tau}^p}\hat{\phi} + \partial^2_{\boldsymbol{\alpha\alpha}}\hat{\psi} : \partial_{\boldsymbol{\beta}}\hat{\phi}] \end{aligned}\right\} \tag{35}$$

within the Eulerian setting. Here we have inserted the canonical evolution equations for the plastic metric and the anisotropy variable. In the case of rate-independent plasticity, the plastic parameter $\lambda$ is given by the loading condition (28). Assuming a plastic loading process with $\lambda > 0$, the plastic parameter can be determined from the so-called consistency condition $\partial_t\hat{\phi} = 0$, which takes—based on the ansatz (22)—the form

$$\begin{aligned} \partial_t \phi &= \partial_{\mathbf{S}^p}\hat{\phi} : \partial_t \mathbf{S}^p + \partial_{\mathbf{B}}\hat{\phi} : \partial_t \mathbf{B} + \partial_{\mathbf{G}^p}\hat{\phi} : \partial_t \mathbf{G}^p \\ &= \partial_{\boldsymbol{\tau}^p}\hat{\phi} : \mathscr{L}_v \boldsymbol{\tau}^p + \partial_{\boldsymbol{\beta}}\hat{\phi} : \mathscr{L}_v \boldsymbol{\beta} + \partial_{\mathbf{c}^p}\hat{\phi} : \mathscr{L}_v \mathbf{c}^p = 0 \end{aligned} \tag{36}$$

in the Lagrangian and Eulerian form. Then the insertion of the rates of the thermodynamic forces (34) and (35) into (36) determines the plastic parameter in the form

$$\begin{aligned} \lambda &= \frac{1}{D}[\partial^2_{\mathbf{S}^p}\hat{\phi} : 2\partial^2_{\mathbf{G}^p\mathbf{C}}\hat{\psi} + \partial_{\mathbf{B}}\hat{\phi} : 2\partial^2_{\mathbf{AC}}\hat{\psi}] : \tfrac{1}{2}\partial_t \mathbf{C} \\ &= \frac{1}{D}[\partial_{\boldsymbol{\tau}^p}\hat{\phi} : 2\partial^2_{\mathbf{c}^p\mathbf{g}}\hat{\psi} + \partial_{\boldsymbol{\beta}}\hat{\phi} : 2\partial^2_{\boldsymbol{\alpha}\mathbf{g}}\hat{\psi}] : \tfrac{1}{2}\mathscr{L}_v \mathbf{g} \end{aligned} \tag{37}$$

with the denominator

$$\begin{aligned} D &:= \partial_{\mathbf{S}^p}\hat{\phi} : [\partial^2_{\mathbf{G}^p\mathbf{G}^p}\hat{\psi} : \partial_{\mathbf{S}^p}\hat{\phi} + \partial^2_{\mathbf{G}^p\mathbf{A}}\hat{\psi} : \partial_{\mathbf{B}}\hat{\phi}] \\ &\quad \partial_{\mathbf{B}}\hat{\phi} : [\partial^2_{\mathbf{AG}^p}\hat{\psi} : \partial_{\mathbf{S}^p}\hat{\phi} + \partial^2_{\mathbf{AA}}\hat{\psi} : \partial_{\mathbf{B}}\hat{\psi}] - \partial_{\mathbf{G}^p}\hat{\phi} : \partial_{\mathbf{S}^p}\hat{\phi} \\ &= \partial_{\boldsymbol{\tau}^p}\hat{\phi} : [\partial^2_{\mathbf{c}^p\mathbf{c}^p}\hat{\psi} : \partial_{\boldsymbol{\tau}^p}\hat{\phi} + \partial^2_{\mathbf{c}^p\boldsymbol{\alpha}}\hat{\psi} : \partial_{\boldsymbol{\beta}}\hat{\phi}] \\ &\quad \partial_{\boldsymbol{\beta}}\hat{\phi} : [\partial^2_{\boldsymbol{\alpha}\mathbf{c}^p}\hat{\psi} : \partial_{\boldsymbol{\tau}^p}\hat{\phi} + \partial^2_{\boldsymbol{\alpha\alpha}}\hat{\psi} : \partial_{\boldsymbol{\beta}}\hat{\phi}] - \partial_{\mathbf{c}^p}\hat{\phi} : \partial_{\boldsymbol{\tau}^p}\hat{\phi}. \end{aligned} \tag{38}$$

The insertion of (37) into $(34)_1$ and $(35)_1$ finally gives the identification of the elastoplastic tangent moduli

$$\boxed{\begin{aligned} \mathbb{C}^{ep} &= 4\partial^2_{\mathbf{CC}}\hat{\psi} - \frac{1}{D}[2\partial^2_{\mathbf{CG}^p}\hat{\psi} : \partial_{\mathbf{S}^p}\hat{\phi} + 2\partial^2_{\mathbf{CA}}\hat{\psi} : \partial_{\mathbf{B}}\hat{\phi}] \\ &\qquad \otimes [\partial_{\mathbf{S}^p}\hat{\phi} : 2\partial^2_{\mathbf{G}^p\mathbf{C}}\hat{\psi} + \partial_{\mathbf{B}}\hat{\phi} : 2\partial^2_{\mathbf{AC}}\hat{\psi}] \\ \mathbb{c}^{ep} &= 4\partial^2_{\mathbf{gg}}\hat{\psi} - \frac{1}{D}[2\partial^2_{\mathbf{gc}^p}\hat{\psi} : \partial_{\boldsymbol{\tau}^p}\hat{\phi} + 2\partial^2_{\mathbf{g}\boldsymbol{\alpha}}\hat{\psi} : \partial_{\boldsymbol{\beta}}\hat{\phi}] \\ &\qquad \otimes [\partial_{\boldsymbol{\tau}^p}\hat{\phi} : 2\partial^2_{\mathbf{c}^p\mathbf{g}}\hat{\psi} + \partial_{\boldsymbol{\beta}}\hat{\phi} : 2\partial^2_{\boldsymbol{\alpha}\mathbf{g}}\hat{\psi}] \end{aligned}} \tag{39}$$

in (33) in the Lagrangian and Eulerian geometric setting. They are characterized by an elastic part and a plastic softening part. Clearly, the latter one is only apparent in the case of plastic loading with $\lambda > 0$.

The symmetry of the elastoplastic tangent moduli (39) for the canonical set of constitutive equations summarized in Table 1 is a key result of the work presented here. Note that this property has been achieved when formulating the flow criterion function (22), which serves in the canonical framework as a plastic potential, in terms of the plastic forces $\mathbf{S}^p$ or $\boldsymbol{\tau}^p$ in connection with the plastic metric $\mathbf{G}^p$ or $\mathbf{c}^p$, respectively. The relationships of this type of flow criterion function to functions formulated in terms of the true stresses $\mathbf{S}$ or $\boldsymbol{\tau}$ in context with the current metric $\mathbf{C}$ or $\mathbf{g}$ is commented on in Section 4. We note that canonical symmetric forms of elastoplastic tangent moduli in multiplicative elastoplasticity have been derived by Miehe (1994a).

### 2.4. *Modifications for non-associative flow response*

In situations where the canonical normal directions of the evolution equations summarized in Table 1 do not fit the real material response under consideration, we modify the constitutive set as follows. Firstly, we consider the third fundamental constitutive function $\hat{\chi}$, referred to in what follows as the plastic potential. It is assumed to depend on the same variable as the flow criterion function (22), i.e.

$$\chi = \hat{\chi}(\mathbf{S}^p, \mathbf{B}; \mathbf{G}^p, \mathbf{G}, X) = \hat{\chi}(\boldsymbol{\tau}^p, \boldsymbol{\beta}; \mathbf{c}^p, \mathbf{c}, x). \tag{40}$$

Then, as a consequence of the material of frame invariance, this function must be an isotropic tensor function of its tensorial arguments. Based on this function, we postulate evolution equations for the plastic metric and the internal variable

$$\left.\begin{aligned} \partial_t \mathbf{G}^p &= \lambda \partial_{\mathbf{S}^p} \hat{\chi} \\ \partial_t \mathbf{A} &= \lambda \partial_{\mathbf{B}} \hat{\chi} \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} \mathscr{L}_v \mathbf{c}^p &= \lambda \partial_{\boldsymbol{\tau}^p} \hat{\chi} \\ \mathscr{L}_v \boldsymbol{\alpha} &= \lambda \partial_{\boldsymbol{\beta}} \hat{\chi} \end{aligned}\right\} \tag{41}$$

which replace the normality rules in Table 1. The plastic loading conditions remain unchanged. Based on these assumptions, the elastoplastic moduli then take the non-symmetric form

$$\begin{aligned} \mathbb{C}^{ep} &= 4\partial^2_{\mathbf{CC}}\hat{\psi} - \frac{1}{D}[2\partial^2_{\mathbf{CG}^p}\hat{\psi} : \partial_{\mathbf{S}^p}\hat{\chi} + 2\partial^2_{\mathbf{CA}}\hat{\psi} : \partial_{\mathbf{B}}\hat{\chi}] \\ &\quad \otimes [\partial_{\mathbf{S}^p}\hat{\phi} : 2\partial^2_{\mathbf{G}^p\mathbf{C}}\hat{\psi} + \partial_{\mathbf{B}}\hat{\phi} : 2\partial^2_{\mathbf{AC}}\hat{\psi}] \\ \mathbb{c}^{ep} &= 4\partial^2_{\mathbf{gg}}\hat{\psi} - \frac{1}{D}[2\partial^2_{\mathbf{gc}^p}\hat{\psi} : \partial_{\boldsymbol{\tau}^p}\hat{\chi} + 2\partial^2_{\mathbf{g}\boldsymbol{\alpha}}\hat{\psi} : \partial_{\mathbf{B}}\hat{\chi}] \\ &\quad \otimes [\partial_{\boldsymbol{\tau}^p}\hat{\phi} : 2\partial^2_{\mathbf{c}^p\mathbf{g}}\hat{\psi} + \partial_{\boldsymbol{\beta}}\hat{\phi} : 2\partial^2_{\boldsymbol{\alpha}\mathbf{g}}\hat{\psi}] \end{aligned} \tag{42}$$

in the Lagrangian and the Eulerian geometric setting in terms of the denominator

$$\begin{aligned} D &:= \partial_{\mathbf{S}^p}\hat{\phi} : [\partial^2_{\mathbf{G}^p\mathbf{G}^p}\hat{\psi} : \partial_{\mathbf{S}^p}\hat{\chi} + \partial^2_{\mathbf{G}^p\mathbf{A}}\hat{\psi} : \partial_{\mathbf{B}}\hat{\chi}] \\ &\quad \partial_{\mathbf{B}}\hat{\phi} : [\partial^2_{\mathbf{AG}^p}\hat{\psi} : \partial_{\mathbf{S}^p}\hat{\chi} + \partial^2_{\mathbf{AA}}\hat{\psi} : \partial_{\mathbf{B}}\hat{\chi}] - \partial_{\mathbf{G}^p}\hat{\phi} : \partial_{\mathbf{S}^p}\hat{\chi} \\ &= \partial_{\boldsymbol{\tau}^p}\hat{\phi} : [\partial^2_{\mathbf{c}^p\mathbf{c}^p}\hat{\psi} : \partial_{\boldsymbol{\tau}^p}\hat{\chi} + \partial^2_{\mathbf{c}^p\boldsymbol{\alpha}}\hat{\psi} : \partial_{\boldsymbol{\beta}}\hat{\chi}] \\ &\quad \partial_{\boldsymbol{\beta}}\hat{\phi} : [\partial^2_{\boldsymbol{\alpha}\mathbf{c}^p}\hat{\psi} : \partial_{\boldsymbol{\tau}^p}\hat{\chi} + \partial^2_{\boldsymbol{\alpha}\boldsymbol{\alpha}}\hat{\psi} : \partial_{\boldsymbol{\beta}}\hat{\chi}] - \partial_{\mathbf{c}^p}\hat{\phi} : \partial_{\boldsymbol{\tau}^p}\hat{\chi}. \end{aligned} \tag{43}$$

## 3. DECOUPLED VOLUMETRIC–ISOCHORIC STRESS RESPONSE

Most materials exhibit a completely different volumetric and isochoric response. A typical example is the case of metal plasticity where the plastic flow is often assumed to be restricted to the isochoric part of the deformation while the volumetric response is assumed to be elastic. Thus, it often makes sense to decompose *a priori* the constitutive response functions into separate volumetric and isochoric contributions. As a consequence, we discuss in this section a geometrically exact decomposition of the set of constitutive elastoplastic equations summarized in Table 1 into decoupled volumetric and isochoric parts. Therefore, we first point out the geometry of a multiplicative split of the tangent map into spherical and unimodular parts. We then set up constitutive sets for decoupled volumetric and isochoric elastoplasticity, respectively, within both the Lagrangian as well as the Eulerian geometric setting.

### 3.1. *Geometry of the volumetric–isochoric decomposition*

The geometrical basis for decoupled volumetric–isochoric constitutive modelling is the multiplicative decomposition of the tangent map $(2)_1$

$$\mathbf{F} = J^{1/3}\bar{\mathbf{F}} \tag{44}$$

into a spherical $J^{1/3}\mathbf{1}$ and a unimodular part $\bar{\mathbf{F}} := J^{-(1/3)}\mathbf{F}$ with the index representations $J^{1/3}\delta^A{}_B$ and $\bar{F}^a{}_A := J^{-(1/3)}F^a{}_A$, respectively. The Jacobian $J := \det[\mathbf{F}(\mathbf{X};t)]$ governs the volumetric part and $\bar{\mathbf{F}}(X;t)$ the isochoric part of the deformation. $\bar{\mathbf{F}} \in SL_3$ is an element of the special linear group $SL_3 := \{\mathbf{T} \mid det[\mathbf{T}] = 1\}$ of unimodular tensors with unit determinant. The decomposition (44) defines locally at $X \in \mathscr{B}$ a new vector space $\bar{T}_X\mathscr{B}$ with dual space $\bar{T}^*_X\mathscr{B}$ associated with the Lagrangian manifold $\mathscr{B}$. We refer to this vector space in what follows as the volumetric intermediate configuration. This geometric viewpoint is visualised in Fig. 3. Then the tangent and normal maps (2) can be split up in a successive format into a part

$$J^{1/3}\mathbf{1} : T_X\mathscr{B} \to \bar{T}_X\mathscr{B} \quad \text{and} \quad J^{-(1/3)}\mathbf{1} : T^*_X\mathscr{B} \to \bar{T}^*_X\mathscr{B} \tag{45}$$

associated with the volumetric part of the deformation and a part

$$\bar{\mathbf{F}} : \bar{T}_X\mathscr{B} \to T_x\mathscr{S} \quad \text{and} \quad \bar{\mathbf{F}}^{-T} : \bar{T}^*_X\mathscr{B} \to T^*_x\mathscr{S} \tag{46}$$

associated with the isochoric part of the deformation.

The volumetric maps (45) transform the Lagrangian variables introduced in (3)–(8) according to

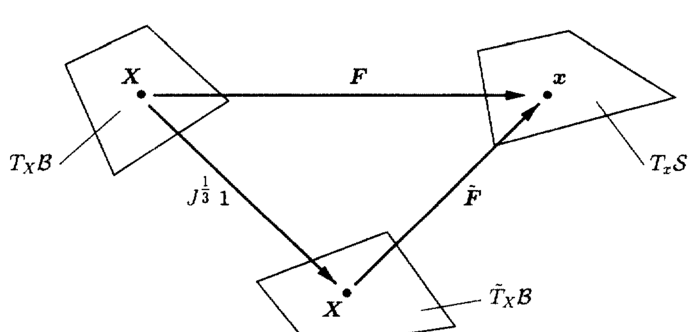


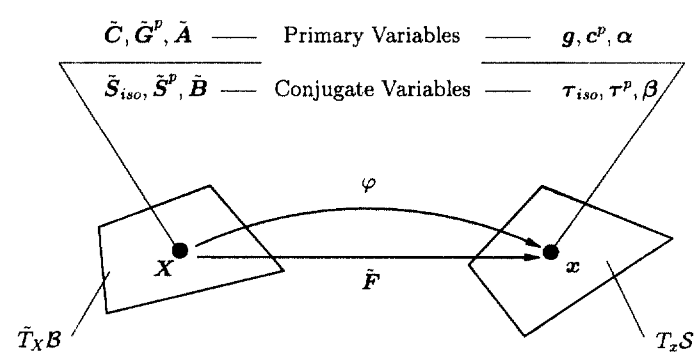

Fig. 3. Volumetric intermediate configuration. The multiplicative decomposition of the tangent map $\mathbf{F} = J^{1/3}\bar{\mathbf{F}}$ into the unimodular part $\bar{\mathbf{F}}$ with $\det[\bar{\mathbf{F}}] = 1$ and the spherical part $J^{1/3}\mathbf{1}$ with $J := \det[\mathbf{F}]$ defines a fictive volumetric intermediate configuration.

$$\left.\begin{aligned}\tilde{\mathbf{C}} &:= J^{-(2/3)}\mathbf{C}\\ \tilde{\mathbf{G}}^p &:= J^{-(2/3)}\mathbf{G}^p\\ \tilde{\mathbf{A}} &:= J^{-(2/3)}\mathbf{A}\end{aligned}\right\}\quad\text{and}\quad\left.\begin{aligned}\tilde{\mathbf{S}} &:= J^{2/3}\mathbf{S}\\ \tilde{\mathbf{S}}^p &:= J^{2/3}\mathbf{S}^p\\ \tilde{\mathbf{B}} &:= J^{2/3}\mathbf{B}\end{aligned}\right\}\tag{47}$$

to the covariant variables $\tilde{\mathbf{C}}(X;t)$, $\tilde{\mathbf{G}}^p(X;t)$, $\tilde{\mathbf{A}}(X;t)$ and the contravariant variables $\tilde{\mathbf{S}}(X;t)$, $\tilde{\mathbf{S}}^p(X;t)$, $\tilde{\mathbf{B}}(X;t)$ with respect to the volumetric intermediate configuration, see Fig. 4. These variables are then transformed by the isochoric maps (46) and the nonlinear deformation map (1) to the Eulerian configuration via

$$\left.\begin{aligned}\mathbf{g} &= \tilde{\mathbf{F}}^{-T}\tilde{\mathbf{C}}\tilde{\mathbf{F}}^{-1}\circ\varphi^{-1}\\ \mathbf{c}^p &= \tilde{\mathbf{F}}^{-T}\tilde{\mathbf{G}}^p\tilde{\mathbf{F}}^{-1}\circ\varphi^{-1}\\ \boldsymbol{\alpha} &= \tilde{\mathbf{F}}^{-T}\tilde{\mathbf{A}}\tilde{\mathbf{F}}^{-1}\circ\varphi^{-1}\end{aligned}\right\}\quad\text{and}\quad\left.\begin{aligned}\boldsymbol{\tau} &= \tilde{\mathbf{F}}\tilde{\mathbf{S}}\tilde{\mathbf{F}}^{T}\circ\varphi^{-1}\\ \boldsymbol{\tau}^p &= \tilde{\mathbf{F}}\tilde{\mathbf{S}}^p\tilde{\mathbf{F}}^{T}\circ\varphi^{-1}\\ \boldsymbol{\beta} &= \tilde{\mathbf{F}}\tilde{\mathbf{B}}\tilde{\mathbf{F}}^{T}\circ\varphi^{-1}\end{aligned}\right\}.\tag{48}$$

The transformations (48) can be viewed as modified pull-back and push-forward operations governed by the nonlinear deformation map (1) and the modified linear tangent and normal maps (46). Denoting with $[\tilde{\cdot}]$ a geometric object with respect to the volumetric intermediate configuration, we introduce the symbolic notation $[\tilde{\cdot}] = \tilde{\varphi}^*(\cdot)$ and $(\cdot) = \tilde{\varphi}_*[\tilde{\cdot}]$ which extend the notation introduced in (9) and (10). In connection with these mappings, we also introduce the modified objective rate $\tilde{\mathscr{L}}_v(\cdot) := \tilde{\varphi}_*\{\partial_t[\tilde{\varphi}^*(\cdot)]\} = \tilde{\varphi}_*\{\partial_t[\tilde{\cdot}]\}$ of spatial objects $(\cdot)$. This rate assumes for second-order tensors the form




| | | | |
|---|---|---|---|
| *Primary Variables:* | $[\tilde{\cdot}]^\flat$ | — | $(\cdot)^\flat$ |
| *Current Metric* | $\tilde{C}$ | — | $g$ |
| *Plastic Metric* | $\tilde{G}^p$ | — | $c^p$ |
| *Anisotropy Variable* | $\tilde{A}$ | — | $\alpha$ |
| *Conjugate Variables:* | $[\tilde{\cdot}]^\sharp$ | — | $(\cdot)^\sharp$ |
| *Stress* | $\tilde{S}_{iso}$ | — | $\tau_{iso}$ |
| *Plastic Force* | $\tilde{S}^p$ | — | $\tau^p$ |
| *Anisotropy Variable* | $\tilde{B}$ | — | $\beta$ |

Fig. 4. Variables associated with isochoric deformation. Primary thermodynamic variables and work-conjugate dual variables on the volumetric intermediate configuration and the Eulerian configuration associated with the isochoric part of the deformation. The Eulerian variable $(\cdot)$ is connected with its dual Lagrangian counterpart $[\tilde{\cdot}]$ by composition with the deformation map $\varphi$ and the modified linear tangent map $\tilde{\mathbf{F}}$ and normal map $\tilde{\mathbf{F}}^{-T}$. That is $(\cdot)^\flat = \tilde{\mathbf{F}}^{-T}[\tilde{\cdot}]^\flat\tilde{\mathbf{F}}^{-1}\circ\varphi^{-1}$ for covariant fields and $(\cdot)^\sharp = \tilde{\mathbf{F}}[\tilde{\cdot}]^\sharp\tilde{\mathbf{F}}^{T}\circ\varphi^{-1}$ for contravariant fields. Objective rates of Eulerian variables are defined by $\tilde{\mathscr{L}}_v(\cdot)^\flat := \tilde{\mathbf{F}}^{-T}\partial_t[\tilde{\cdot}]^\flat\tilde{\mathbf{F}}^{-1}\circ\varphi^{-1}$ and $\tilde{\mathscr{L}}_v(\cdot)^\sharp := \tilde{\mathbf{F}}\partial_t[\tilde{\cdot}]^\sharp\tilde{\mathbf{F}}^{T}\circ\varphi^{-1}$, respectively. The reference metric of the volumetric intermediate configuration is denoted with $\mathbf{G}$ and has the Eulerian form $\tilde{\mathbf{c}} = \tilde{\mathbf{F}}^{-T}\mathbf{G}\tilde{\mathbf{F}}^{-1}\circ\varphi^{-1}$.

$$\left.\begin{aligned} \tilde{\mathscr{L}}_v(\cdot)^b &= \overset{\bullet}{\overline{(\cdot)}}{}^b + \tilde{\mathbf{l}}^T(\cdot)^b + (\cdot)^b\tilde{\mathbf{l}} \\ \tilde{\mathscr{L}}_v(\cdot)^\# &= \overset{\bullet}{\overline{(\cdot)}}{}^\# - \tilde{\mathbf{l}}(\cdot)^\# - (\cdot)^\#\tilde{\mathbf{l}}^T \end{aligned}\right\} \tag{49}$$

for covariant fields and contravariant fields, see also Fig. 4. Here $\tilde{\mathbf{l}}$ is the part of the spatial velocity gradient associated with the isochoric part of the deformation and appears in the additive split

$$\mathbf{l} = \partial_t J J^{-1}\tfrac{1}{3}\mathbf{1} \circ \varphi^{-1} + \tilde{\mathbf{l}} \quad \text{with} \quad \tilde{\mathbf{l}} := \partial_t \tilde{\mathbf{F}}\tilde{\mathbf{F}}^{-1} \circ \varphi^{-1} \tag{50}$$

of the velocity gradient (12) into spherical and deviatoric parts.

### 3.2. *Decoupling of the constitutive response*

The basis for the decomposition of the constitutive response functions is an assumed split of the free energy function (13) into decoupled volumetric and isochoric contributions

$$\hat{\psi} = \hat{\psi}_{\text{vol}} + \hat{\psi}_{\text{iso}}. \tag{51}$$

The functional dependencies of the functions $\hat{\psi}_{\text{vol}}$ and $\hat{\psi}_{\text{iso}}$ are investigated in the two subsections below. The split (51) is consistent with an additive decomposition of the stresses into spherical and deviatoric contributions

$$\boxed{\tilde{\mathbf{S}} = P\tilde{\mathbf{C}}^{-1} + \tilde{\mathbf{S}}_{\text{iso}} \quad \text{and} \quad \boldsymbol{\tau} = P\mathbf{g}^{-1} + \boldsymbol{\tau}_{\text{iso}}.} \tag{52}$$

Here, $P$ is the volumetric stress contribution, i.e. the negative pressure. The isochoric contribution to the stresses must be traceless with respect to the current metric, i.e. $\tilde{\mathbf{S}}_{\text{iso}} : \tilde{\mathbf{C}} = 0$ and $\boldsymbol{\tau}_{\text{iso}} : \mathbf{g} = 0$. Thus, $\tilde{\mathbf{S}}_{\text{iso}}$ and $\boldsymbol{\tau}_{\text{iso}}$ are deviators with respect to the current metric

$$\tilde{\mathbf{S}}_{\text{iso}} = \text{dev}_{\tilde{\mathbf{C}}}[\tilde{\mathbf{S}}] \quad \text{and} \quad \boldsymbol{\tau}_{\text{iso}} = \text{dev}_{\mathbf{g}}[\boldsymbol{\tau}]. \tag{53}$$

We use the notation $\text{dev}_{\tilde{\mathbf{C}}}[\tilde{\cdot}]$ and $\text{dev}_{\mathbf{g}}(\cdot)$ for deviator operators with respect to the metric $\tilde{\mathbf{C}}$ and $\mathbf{g}$, respectively. They have the typical structure $\text{dev}_{\mathbf{g}}(\cdot)^b := (\cdot)^b - \frac{1}{3}[(\cdot)^b : \mathbf{g}^{-1}]\mathbf{g}$ and $dev_{\mathbf{g}}(\cdot)^\# := (\cdot)^b - \frac{1}{3}[(\cdot)^b : \mathbf{g}]\mathbf{g}^{-1}$ when applied to covariant and contravariant second-order tensors.

The insertion of the split (52) into the Clausius–Planck inequality (17) results, in connection with the assumption (51), in an additive decomposition of the dissipation into volumetric and isochoric contributions

$$\mathscr{D}^p = \mathscr{D}^p_{\text{vol}} + \mathscr{D}^p_{\text{iso}} \geqslant 0. \tag{54}$$

Because both parts of the deformation are *a priori* assumed to be decoupled and therefore, independent, each contribution to the dissipation must be positive. This gives the thermodynamic restrictions

$$\mathscr{D}^p_{\text{vol}} := P\partial_t J J^{-1} - \partial_t \hat{\psi}_{\text{vol}} \geqslant 0 \tag{55}$$

for the volumetric part of the deformation and

$$\mathscr{D}^p_{\text{iso}} := \tilde{\mathbf{S}}_{\text{iso}} : \tfrac{1}{2}\tilde{\mathscr{L}}_v\tilde{\mathbf{C}} - \partial_t\hat{\psi}_{\text{iso}} = \boldsymbol{\tau}_{\text{iso}} : \tfrac{1}{2}\mathscr{L}_v\mathbf{g} - \partial_t\hat{\psi}_{\text{iso}} \geqslant 0 \tag{56}$$

for the isochoric part of the dissipation in a geometric setting relative to the intermediate configuration in terms of the rate $\tilde{\mathscr{L}}_v\tilde{\mathbf{C}} := J^{-(2/3)}\partial_t\mathbf{C}$ and relative to the current configuration. In the subsequent development we exploit both mechanisms separately.

Table 2. Constitutive set of volumetric elastoplasticity

| | |
|---|---|
| *Free energy* | $\psi_{\mathrm{vol}} = \hat{\psi}_{\mathrm{vol}}(J, A\,; J^p, X)$ |
| Stresses | $P = J\partial_J\hat{\psi}_{\mathrm{vol}}$ |
| Plastic force | $P^p := -\partial_{J^p}\hat{\psi}_{\mathrm{vol}}$ |
| Internal force | $B := -\partial_A\hat{\psi}_{\mathrm{vol}}$ |
| *Flow criterion* | $\phi_{\mathrm{vol}} = \hat{\phi}_{\mathrm{vol}}(P^p, B\,; J^p, X)$ |
| Flow rule | $\partial_t J^p = \mu\partial_{P^p}\hat{\phi}_{\mathrm{vol}}$ |
| Evolution | $\partial_t A = \mu\partial_B\hat{\phi}_{\mathrm{vo}}$ |
| Loading | $\mu \geqslant 0\,;\ \hat{\phi}_{\mathrm{vol}} \leqslant 0\,;\ \mu\hat{\phi}_{\mathrm{vol}} = 0$ |
| [Viscoplastic | $\mu := (1/\eta_{\mathrm{vol}})\hat{p}'_{\mathrm{vol}}(\phi^{+}_{\mathrm{vol}})]$ |

### 3.3. *Decoupled volumetric constitutive response*

The volumetric response is per definition isotropic and governed by the Jacobian $J$ in (44). We take into account possibly elastoplastic volumetric behavior and introduce the plastic variable $J^p(X;t)$ and the isotropic hardening variable $A(X;t)$ with some initial conditions, for example $J^p(X;t_0) = 1$ and $A(X;t_0) = 0$. Then

$$\boxed{\psi_{\mathrm{vol}} = \hat{\psi}_{\mathrm{vol}}(J, A\,; J^p, X)} \tag{57}$$

is an ansatz for the volumetric free energy in (51). Insertion of its evolution into the dissipation inequality (55) then yields

$$\mathscr{D}^p_{\mathrm{vol}} = [P\partial_t JJ^{-1} - \partial_J\hat{\psi}_{\mathrm{vol}}]\partial_t J - \partial_{J^p}\hat{\psi}_{\mathrm{vol}}\partial_t J^p - \partial_A\hat{\psi}_{\mathrm{vol}}\partial_t A \geqslant 0. \tag{58}$$

An argumentation similar to that in Section 2.2.1 yields the constitutive functions for the volumetric stress in Table 2 and leaves the reduced dissipation function

$$\hat{\mathscr{D}}^p_{\mathrm{vol}} = P^p\partial_t J^p + B\partial_t A \geqslant 0 \tag{59}$$

with the volumetric plastic force $P^p(X;t)$ and internal force $B(X;t)$ defined in Table 1. We consider an elastic domain $\mathbb{E}_{\mathrm{vol}} := \{(P^p, B) \in \mathbb{R} \times \mathbb{R} \mid \hat{\phi}_{\mathrm{vol}}(P^p, B\,; J^p, X) \leqslant 0\}$ in the space of the volumetric thermodynamic forces governed by the volumetric yield criterion function

$$\boxed{\phi_{\mathrm{vol}} = \hat{\phi}_{\mathrm{vol}}(P^p, B\,; J^p, X).} \tag{60}$$

The canonical structure of the evolution equations for the volumetric plastic deformation and the volumetric internal variable follows from the argument

$$\hat{\mathscr{D}}^p_{\mathrm{vol}} - \mathscr{D}^p_{\mathrm{vol}}{}^* := [P^p - P^{p*}] : \partial_t J^p + [B - B^*] : \partial_t A \geqslant 0 \tag{61}$$

for all admissible variations $(P^{p*}, B^*) \in \mathbb{E}_{\mathrm{vol}}$ of the volumetric thermodynamic forces. A procedure similar to that in Section 2.2.2 yields the evolution equations in Table 2 including the loading–unloading conditions for the volumetric plastic parameter $\mu$.

### 3.4. *Decoupled isochoric constitutive response*

Let $\hat{\psi}_{\mathrm{iso}}$ denote the change in locally stored free energy during the isochoric part of the deformation process from the volumetric intermediate configuration to the current configuration. We assume a functional dependence on the current metric, the anisotropy variable, the plastic metric and the reference metric of the volumetric intermediate configuration. Consider the particular form

$$\boxed{\psi = \hat{\psi}_{\mathrm{iso}}(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}\,; \tilde{\mathbf{G}}^p, \mathbf{G}, X) = \hat{\psi}_{\mathrm{iso}}(\mathbf{g}, \boldsymbol{\alpha}\,; \mathbf{c}^p, \tilde{\mathbf{c}}, x)} \tag{62}$$

where we have assumed similar to (13) an identical function $\hat{\psi}$ in the setting relative to the volumetric intermediate configuration and the Eulerian geometric setting. Then the principle of material frame invariance restricts $\hat{\psi}_{\text{iso}}$ to an isotropic tensor function with

$$\hat{\psi}_{\text{iso}}(\mathbf{QgQ}^T, \mathbf{Q}\boldsymbol{\alpha}\mathbf{Q}^T; \mathbf{Qc}^p\mathbf{Q}^T, \mathbf{Q}\tilde{\mathbf{c}}\mathbf{Q}^T, x) = \hat{\psi}_{\text{iso}}(\mathbf{g}, \boldsymbol{\alpha}; \mathbf{c}^p, \tilde{\mathbf{c}}, x) \tag{63}$$

and

$$\hat{\psi}_{\text{iso}}(\mathbf{Q}\tilde{\mathbf{C}}\mathbf{Q}^T, \mathbf{Q}\tilde{\mathbf{A}}\mathbf{Q}^T; \mathbf{Q}\tilde{\mathbf{G}}^p\mathbf{Q}^T, \mathbf{QGQ}^T, X) = \hat{\psi}_{\text{iso}}(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}; \tilde{\mathbf{G}}^p, \mathbf{G}, X) \tag{64}$$

for all rotations $\mathbf{Q} \in SO_3$, see also the discussion in Section 2.2.1. Insertion of the evolution of (62) into the dissipation inequality (56) yields

$$\begin{aligned} \mathscr{D}^p_{\text{iso}} &= [\tilde{\mathbf{S}}_{\text{iso}} - \operatorname{dev}_{\tilde{\mathbf{C}}}[2\partial_{\tilde{\mathbf{C}}}\hat{\psi}_{\text{iso}}]] : \tfrac{1}{2}\mathscr{L}_v\tilde{\mathbf{C}} - \partial_{\tilde{\mathbf{G}}^p}\hat{\psi}_{\text{iso}} : \partial_t\tilde{\mathbf{G}}^p - \partial_{\tilde{\mathbf{A}}}\hat{\psi}_{\text{iso}} : \partial_t\tilde{\mathbf{A}} \\ &= [\boldsymbol{\tau}_{\text{iso}} - \operatorname{dev}_{\mathbf{g}}[2\partial_{\mathbf{g}}\hat{\psi}_{\text{iso}}]] : \tfrac{1}{2}\mathscr{L}_v\mathbf{g} - \partial_{\mathbf{c}^p}\hat{\psi}_{\text{iso}} : \tilde{\mathscr{L}}_v\mathbf{c}^p - \partial_{\boldsymbol{\alpha}}\hat{\psi}_{\text{iso}} : \tilde{\mathscr{L}}_v\boldsymbol{\alpha} \geqslant 0. \end{aligned} \tag{65}$$

An argument similar to that in Section 2.2.1 then gives the constitutive expressions for the stresses in Table 3. Introducing per definition the plastic force $\tilde{\mathbf{S}}^p$ and internal force $\tilde{\mathbf{B}}$ in Table 3, the reduced dissipation inequality associated with the isochoric part of the deformation takes the form

$$\hat{\mathscr{D}}^p_{\text{iso}} = \tilde{\mathbf{S}}^p : \partial_t\tilde{\mathbf{G}}^p + \tilde{\mathbf{B}} : \partial_t\tilde{\mathbf{A}} = \boldsymbol{\tau}^p : \tilde{\mathscr{L}}_v\mathbf{c}^p + \boldsymbol{\beta} : \tilde{\mathscr{L}}_v\boldsymbol{\alpha} \geqslant 0. \tag{66}$$

Observe carefully that we have formulated the function in terms of rates related to the isochoric part of the deformation only, i.e. governed by the isochoric maps (46). We consider elastic domains $\mathbb{E}^L_{\text{iso}} := \{(\tilde{\mathbf{S}}^p, \tilde{\mathbf{B}}) \in \mathbb{R}^5 \times \mathbb{R}^5 \mid \hat{\phi}_{\text{iso}}(\tilde{\mathbf{S}}^p, \tilde{\mathbf{B}}; \tilde{\mathbf{G}}^p, \mathbf{G}, X) \leqslant 0\}$ and $\mathbb{E}^E_{\text{iso}} := \{(\boldsymbol{\tau}^p, \boldsymbol{\beta}) \in \mathbb{R}^5 \times \mathbb{R}^5 \mid \hat{\phi}_{\text{iso}}(\boldsymbol{\tau}, \boldsymbol{\beta}; \mathbf{c}^p, \tilde{\mathbf{c}}, x) \leqslant 0\}$ in the space of the isochoric thermodynamic forces governed by the isochoric yield criterion function

$$\boxed{\phi_{\text{iso}} = \hat{\phi}_{\text{iso}}(\tilde{\mathbf{S}}^p, \tilde{\mathbf{B}}; \tilde{\mathbf{G}}^p, \mathbf{G}, X) = \hat{\phi}_{\text{iso}}(\boldsymbol{\tau}^p, \boldsymbol{\beta}; \mathbf{c}^p, \tilde{\mathbf{c}}, x)} \tag{67}$$

assuming the identical function $\hat{\psi}$ in the setting relative to the volumetric intermediate configuration and the Eulerian geometric setting in analogy to (22). Then the principle of material frame invariance restricts the admissible functional structure (62) to an isotropic tensor function with

$$\hat{\phi}_{\text{iso}}(\mathbf{Q}\boldsymbol{\tau}^p\mathbf{Q}^T, \mathbf{Q}\boldsymbol{\beta}\mathbf{Q}^T; \mathbf{Qc}^p\mathbf{Q}^T, \mathbf{Q}\tilde{\mathbf{c}}\mathbf{Q}^T, x) = \hat{\phi}_{\text{iso}}(\boldsymbol{\tau}^p, \boldsymbol{\beta}; \mathbf{c}^p, \tilde{\mathbf{c}}, x) \tag{68}$$

and

Table 3. Constitutive set of isochoric anisotropic elastoplasticity

| | Lagrangian setting | Eulerian setting |
|---|---|---|
| *Free energy* | $\psi_{\text{iso}} = \hat{\psi}_{\text{iso}}(\tilde{\mathbf{C}}, \tilde{\mathbf{A}}; \tilde{\mathbf{G}}^p, \mathbf{G}, X)$ | $\psi_{\text{iso}} = \hat{\psi}_{\text{iso}}(\mathbf{g}, \boldsymbol{\alpha}; \mathbf{c}^p, \tilde{\mathbf{c}}, x)$ |
| Stresses | $\tilde{\mathbf{S}}_{\text{iso}} = \operatorname{dev}_{\tilde{\mathbf{C}}}[2\partial_{\tilde{\mathbf{C}}}\hat{\psi}_{\text{iso}}]$ | $\boldsymbol{\tau}_{\text{iso}} = \operatorname{dev}_{\mathbf{g}}[2\partial_{\mathbf{g}}\hat{\psi}_{\text{iso}}]$ |
| Plastic force | $\tilde{\mathbf{S}}^p := -\partial_{\tilde{\mathbf{G}}^p}\hat{\psi}_{\text{iso}}$ | $\boldsymbol{\tau}^p := -\partial_{\mathbf{c}^p}\hat{\psi}_{\text{iso}}$ |
| Internal force | $\tilde{\mathbf{B}} := -\partial_{\tilde{\mathbf{A}}}\hat{\psi}_{\text{iso}}$ | $\boldsymbol{\beta} := -\partial_{\boldsymbol{\alpha}}\hat{\psi}_{\text{iso}}$ |
| *Yield function* | $\phi_{\text{iso}} = \hat{\phi}_{\text{iso}}(\tilde{\mathbf{S}}^p, \tilde{\mathbf{B}}; \tilde{\mathbf{G}}^p, \mathbf{G}, X)$ | $\phi_{\text{iso}} = \hat{\phi}_{\text{iso}}(\boldsymbol{\tau}^p, \boldsymbol{\beta}; \mathbf{c}^p, \tilde{\mathbf{c}}, x)$ |
| Flow rule | $\partial_t\tilde{\mathbf{G}}^p = v\partial_{\tilde{\mathbf{S}}^p}\hat{\phi}_{\text{iso}}$ | $\tilde{\mathscr{L}}_v\mathbf{c}^p = v\partial_{\boldsymbol{\tau}^p}\hat{\phi}_{\text{iso}}$ |
| Evolution | $\partial_t\tilde{\mathbf{A}} = v\partial_{\tilde{\mathbf{B}}}\hat{\phi}_{\text{iso}}$ | $\tilde{\mathscr{L}}_v\boldsymbol{\alpha} = v\partial_{\boldsymbol{\beta}}\hat{\phi}_{\text{iso}}$ |
| Loading | $v \geqslant 0;\ \hat{\phi}_{\text{iso}} \leqslant 0;\ v\hat{\phi}_{\text{iso}} = 0$ | $v \geqslant 0;\ \hat{\phi}_{\text{iso}} \leqslant 0;\ v\hat{\phi}_{\text{iso}} = 0$ |
| [Viscoplastic | $v := (1/\eta_{\text{iso}})\hat{p}'_{\text{iso}}(\phi^-_{\text{iso}})$ | $v := (1/\eta_{\text{iso}})\hat{p}'_{\text{iso}}(\phi^+_{\text{iso}})]$ |

$$\hat{\phi}_{\text{iso}}(\mathbf{Q}\tilde{\mathbf{S}}^p\mathbf{Q}^T, \mathbf{Q}\tilde{\mathbf{B}}\mathbf{Q}^T; \mathbf{Q}\tilde{\mathbf{G}}^p\mathbf{Q}^T, \mathbf{Q}\mathbf{G}\mathbf{Q}^T, X) = \hat{\phi}_{\text{iso}}(\tilde{\mathbf{S}}^p, \tilde{\mathbf{B}}; \tilde{\mathbf{G}}^p, \mathbf{G}, X) \tag{69}$$

for all rotations $\mathbf{Q} \in SO_3$. The canonical form of the evolution equation for the plastic deformation and the internal variable associated with the isochoric part of the deformation follows from the argument

$$\begin{aligned} \hat{\mathscr{D}}^p_{\text{iso}} - \mathscr{D}^{p*}_{\text{iso}} &:= [\tilde{\mathbf{S}}^p - \tilde{\mathbf{S}}^{p*}] : \partial_t \tilde{\mathbf{G}}^p + [\tilde{\mathbf{B}} - \tilde{\mathbf{B}}^*] : \partial_t \tilde{\mathbf{A}} \\ &= [\boldsymbol{\tau}^p - \boldsymbol{\tau}^{p*}] : \tilde{\mathscr{L}}_v \mathbf{c}^p + [\boldsymbol{\beta} - \boldsymbol{\beta}^*] : \tilde{\mathscr{L}}_v \boldsymbol{\alpha} \geqslant 0 \end{aligned} \tag{70}$$

for all admissible variations $(\tilde{\mathbf{S}}^{p*}, \tilde{\mathbf{B}}^*) \in \mathbb{E}^L_{\text{iso}}$ and $(\boldsymbol{\tau}^{p*}, \boldsymbol{\beta}^*) \in \mathbb{E}^E_{\text{iso}}$ of the isochoric thermodynamic forces. This yields the evolution equations in Table 3 including the loading–unloading conditions for the isochoric plastic parameter $\nu$.

## 4. ELASTIC STRAINS AND DRIVING STRESSES

The framework of elastoplasticity outlined in the two Sections above is still very general. Recall that the free energy function (13) is formulated in terms of the current metric and the plastic metric in a completely general context. This covers a wide range of approaches for the constitutive description of the local elastic response of an elastoplastic solid. In this Section we constrict this general framework by considering some particular definitions of elastic strain tensors. They constitute *a priori* a relationship between the current metric and the plastic metric which is assumed to enter the free energy function. Recall furthermore, that the yield criterion function (22), which serves within the canonical framework as a plastic potential for the plastic increments, has been formulated in terms of the plastic force and the plastic metric. This provides several possibilities for the formulation of the local plastic response. We therefore, constrict the possible settings to some particular applications by the introduction of driving stress tensors. These stress tensors constitute *a priori* a relationship between the plastic forces and the plastic metric which is assumed to enter the yield criterion function.

### 4.1. *Particular formulations of elastic strain tensors*

Consider the particular form of the isotropic free energy function (13)

$$\psi = \hat{\psi}(\mathbf{E}; \mathbf{G}, X) = \hat{\psi}(\mathbf{e}; \mathbf{c}, x) \tag{71}$$

where the tensor fields $\mathbf{E}(X; t)$ on $\mathscr{B}$ and $\mathbf{e}(x; t)$ on $\mathscr{S}$ denote Lagrangian and Eulerian elastic strain tensors, respectively. Three examples for the definition of these tensors are

$$\left.\begin{aligned} \mathbf{E}_1 &:= \mathbf{C} - \mathbf{G}^p \\ \mathbf{E}_2 &:= \mathbf{G}^{p-1}\mathbf{C}\mathbf{G}^{p-1} \\ \mathbf{E}_3 &:= \mathbf{C}\mathbf{G}^{p-1} \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} \mathbf{e}_1 &:= \mathbf{g} - \mathbf{c}^p \\ \mathbf{e}_2 &:= \mathbf{c}^{p-1}\mathbf{g}\mathbf{c}^{p-1} \\ \mathbf{e}_3 &:= \mathbf{g}\mathbf{c}^{p-1} \end{aligned}\right\}. \tag{72}$$

The first approach relates the current metric to the plastic metric in an additive format similar to the geometric linear theory of elastoplasticity. The further two definitions relate both tensors in a multiplicative format. The *a priori* definitions (72) of particular combinations of the current metric and the plastic metric induce vice verse relationships between their conjugate thermodynamic variables. In particular, the three definitions (72) induce the representations of the plastic forces

$$\left.\begin{array}{l} \mathbf{S}_1^p := \frac{1}{2}\mathbf{S} \\ \mathbf{S}_2^p := \operatorname{sym}[\mathbf{G}^{p-1}\mathbf{C}\mathbf{S}] \\ \mathbf{S}_3^p := \frac{1}{2}\mathbf{G}^{p-1}\mathbf{C}\mathbf{S} \end{array}\right\} \quad \text{and} \quad \left.\begin{array}{l} \boldsymbol{\tau}_1^p := \frac{1}{2}\boldsymbol{\tau} \\ \boldsymbol{\tau}_2^p := \operatorname{sym}[\mathbf{c}^{p-1}\mathbf{g}\boldsymbol{\tau}] \\ \boldsymbol{\tau}_3^p := \frac{1}{2}\mathbf{c}^{p-1}\mathbf{g}\boldsymbol{\tau} \end{array}\right\}. \tag{73}$$

This result follows in a straightforward manner by exploitation of the constitutive function for the plastic forces in Table 1 where the last formulation is restricted to isotropic response and has been obtained only by exploitation of commutative properties of isotropic tensor functions. In the first approach, the plastic force is simply proportional to the stress as in the geometric linear theory. The second formulation defines the plastic force in a multiplicative format as the symmetric part of the mixed-variant stress $\mathbf{CS}$ and $\mathbf{g}\boldsymbol{\tau}$, respectively. Note that these mixed-variant stresses arise naturally as the thermodynamic forces of nine-dimensional plasticity based on the multiplicative decomposition of the tangent map $\mathbf{F} = \mathbf{F}^e\mathbf{F}^p$ into elastic and plastic parts, see e.g. Mandel (1972), Le and Stumpf (993), Miehe (1994a, 1996a), among others. Here $\mathbf{F}^p$ is a possibly non-symmetric second-order tensor, which includes, according to its polar decomposition $\mathbf{F}^p = \mathbf{R}^p\mathbf{U}^p$ a possible local plastic rotation $\mathbf{R}^p \in SO_3$. However, within the formulation of six-dimensional plasticity proposed here, the mixed-variant stresses $\mathbf{CS}$ and $\mathbf{g}\boldsymbol{\tau}$ are symmetrized by raising the first index with the plastic metric. Thus, the second approach in $(72)_2$ is consistent with a formulation of multiplicative elastoplasticity with *a priori* assumption $\mathbf{F} = \mathbf{F}^e\mathbf{U}^p$, i.e. postulating $\mathbf{R}^p = \mathbf{1}$ and identifying the plastic metric $\mathbf{G}^p = \mathbf{U}^p$ with the plastic stretch $\mathbf{U}^p \in \mathrm{sym}_3^+$, see e.g. Haupt (1985) and Lubliner (1990) for a discussion of the restrictions of this approach. In view to a further extended discussion we refer in this context to the works Schieck and Stumpf (1993, 1995), where a unique decomposition $\mathbf{F} = \mathbf{R}\mathbf{U}^e\mathbf{U}^p$ without the assumption $\mathbf{R}^p = \mathbf{1}$ is proposed. In the formulation proposed here, $(72)_2$ defines the elastic response with respect to a fictive plastic intermediate configuration which is defined locally at $X \in \mathscr{B}$ by splitting up the tangent map $(2)_1$ into the plastic stretch $\mathbf{G}^p = \mathbf{U}^p$ and the part $\mathbf{F}^e$, where the latter describes the elastic stretch and the local rigid body rotations. The third approach in $(72)_3$ can be viewed as a short cut convenient representation of the second approach valid only in the isotropic case where anisotropy variables are not apparent.

Both formulations have a very important property with regard to the description of isochoric elastoplastic response as outline in Section 3. In this case, the stresses $\mathbf{S}$ and $\boldsymbol{\tau}$ are deviators with respect to the current metric $\mathbf{C}$ and $\mathbf{g}$, respectively, see Table 3. Then a few algebraic manipulations show that the expressions $(73)_2$ and $(73)_3$ transform the deviators with respect to the current metric to deviators with respect to the plastic metric. Thus, if the stresses are deviatoric with respect to the current metric, then the plastic forces are also deviatoric but with respect to the plastic metric. This property is crucial with regard to the modelling of isochoric flow response for plasticity models formulated in terms of the stresses and the current metric considered below.

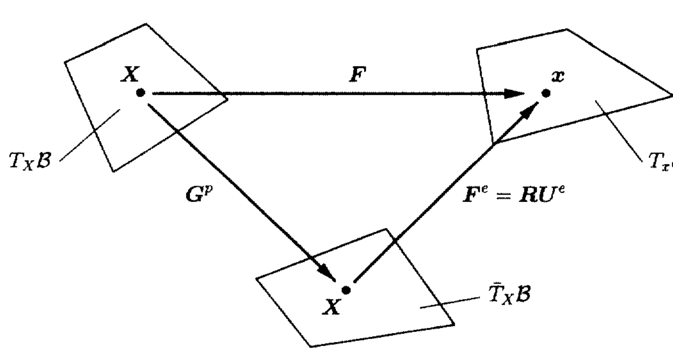



Fig. 5. A fictive plastic intermediate configuration. The tangent map $\mathbf{F}: T_X\mathscr{B} \to T_x\mathscr{S}$ is split up $\mathbf{F} = \mathbf{F}^e\mathbf{G}^p = (\mathbf{R}\mathbf{U}^e)\mathbf{G}^p$ into plastic stretch, elastic stretch and local rotation. Then the plastic stretch $\mathbf{G}^p$ defines locally at $X \in \mathscr{B}$ a fictive plastic intermediate configuration.

### 4.2. *Particular formulations of driving stress tensors*

The next important point is the formulation of the yield criterion function (22) which also serves within the canonical approach as the plastic potential for the constitutive rate equations. Similar to the consideration outlined above we discuss here model problems of the particular form

$$\phi = \hat{\phi}(\boldsymbol{\Sigma};\mathbf{G},X) = \hat{\phi}(\boldsymbol{\sigma};\mathbf{c},x) \tag{74}$$

for the case of ideal plasticity. We refer to the tensor fields $\boldsymbol{\Sigma}(\mathbf{X};t)$ on $\mathscr{B}$ and $\boldsymbol{\sigma}(x;t)$ on $\mathscr{S}$ as the Lagrangian and Eulerian driving stress tensors, which drive the local plastic deformation within the elastoplastic solid. We consider here three different definitions

$$\left.\begin{aligned} \boldsymbol{\Sigma}_1 &:= 2\mathbf{S}^p = \mathbf{S} \\ \boldsymbol{\Sigma}_2 &:= \mathbf{G}^p\mathbf{S}^p\mathbf{G}^p = \operatorname{sym}[\mathbf{CSG}^p] \\ \boldsymbol{\Sigma}_3 &:= 2\mathbf{G}^p\mathbf{S}^p = \mathbf{CS} \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} \boldsymbol{\sigma}_1 &:= 2\boldsymbol{\tau}^p = \boldsymbol{\tau} \\ \boldsymbol{\sigma}_2 &:= \mathbf{c}^p\boldsymbol{\tau}^p\mathbf{c}^p = \operatorname{sym}[\mathbf{g}\boldsymbol{\tau}\mathbf{c}^p] \\ \boldsymbol{\sigma}_3 &:= 2\mathbf{c}^p\boldsymbol{\tau}^p = \mathbf{g}\boldsymbol{\tau} \end{aligned}\right\} \tag{75}$$

which characterize different plastic flow responses.

Note that we have inserted the relationships (73) for the plastic forces. Thus, we relate, although not in any cases necessary, the three choices of the driving stresses (75) to the three choices (72) of the elastic strains. The first approach simply says that the driving stress is identical with the plastic force. This constitutes a plasticity model based on flow criterion functions formulated in terms of the stresses and the reference metric of the Lagrangian configuration. This is in many cases not acceptable where one wants to formulate the yield criterion in terms of the stresses and the current metric. Considering $(75)_{2,3}$ one realizes that the second and third approach does exactly this, at least for the case of isotropic response. The plastic driving stress of the third approach is the mixed-variant stress $\mathbf{CS}$ and $\mathbf{g}\boldsymbol{\tau}$ which has also been used in multiplicative elastoplasticity, as already mentioned above. However, we have a fundamental difference to nine-dimensional multiplicative elastoplasticity in the case of anisotropic response where these tensors are symmetrized with respect to the plastic metric, see $(73)_2$. As mentioned above, in the case of isochoric plasticity the plastic forces $(73)_{2,3}$ are deviators with respect to the plastic metric. As a consequence, the flow rules in Table 3 preserve exactly the plastic volume in this situation. That means the plastic metric $\tilde{\mathbf{G}}^p \in SL_3$ and $\tilde{\mathbf{c}}^p \in SL_3$ are elements of the special linear group $SL_3$ with unit determinant, i.e.

$$\det[\tilde{\mathbf{G}}^p] = 1 \quad \text{and} \quad \det[\mathbf{c}^p] = 1 \tag{76}$$

for all times of the isochoric deformation process. This property is often referred to as the plastic incompressibility constraint.

In the following Sections we consider particular formulations for isotropic and anisotropic elastoplastic response. Therefore, we restrict the consideration to plasticity models with yield criterion functions formulated in terms of the stresses and the current metric. For the description of isotropic response we consider the third approach to elastic strain tensors and plastic driving forces as the most convenient one. For the description of anisotropic response in terms of anisotropy tensors we choose the second approach to elastic strains and plastic driving stresses discussed here.

## 5. APPLICATION TO ISOTROPIC ELASTOPLASTIC RESPONSE

We investigate now the specification of the constitutive equations proposed in Section 2 to the case of isotropic elastoplasticity where anisotropy tensors of the type (5) and (8) are not present. The particular model considered here is based on the definitions $(72)_3$ of elastic strains and $(75)_3$ of driving stresses and covers therefore, a model of elastoplasticity formulated in terms of the stresses and the current metric. We set up first a formulation in terms of invariants of the current metric and the plastic force and consider then a formulation in terms of elastic principal stretches and principal stresses.

5.1. *Isotropic elastoplasticity in terms of invariants*

For isotropic elastoplastic response we assume a dependence of the free energy on the invariants $\{I_i\}_{i=1,3}$ of the current metric and a generic scalar variable $A$ which describes for instance, isotropic hardening phenomena or isotropic damage effects. Thus, we consider the particular form

$$\boxed{\psi = \hat{\psi}(I_1, I_2, I_3, A)} \tag{77}$$

of (13). The invariants $\{I_i\}_{i=1,3}$ of the current metric are evaluated with respect to the plastic metric, which is assumed to play the role of a reference metric for the partial elastic response. Within this context we introduce the ground invariants

$$\left.\begin{array}{l} I_1 := \mathrm{tr}\,[\mathbf{E}] \\ I_2 := \mathrm{tr}\,[\mathbf{E}^2]/2 \\ I_3 := \mathrm{tr}\,[\mathbf{E}^3]/3 \end{array}\right\} \quad \text{or} \quad \left.\begin{array}{l} I_1 := \mathrm{tr}\,[\mathbf{e}] \\ I_2 := \mathrm{tr}\,[\mathbf{e}^2]/2 \\ I_3 := \mathrm{tr}\,[\mathbf{e}^3]/3 \end{array}\right\} \tag{78}$$

in terms of the Lagrangian and Eulerian mixed-variant elastic strain tensors

$$\mathbf{E} := \mathbf{C}\mathbf{G}^{p-1} \quad \text{and} \quad \mathbf{e} := \mathbf{g}\mathbf{c}^{p-1} \tag{79}$$

which have already been introduced in $(72)_3$. The covariant–contravariant tensor fields $\mathbf{E}(X;t)$ and $\mathbf{e}(x;t)$ have been denoted in Lehmann (1972) and Miehe (1994c) as metric transformation tensors. Taking into account the derivatives $\partial_{\mathbf{T}}\,\mathrm{tr}\,[\mathbf{T}] = \mathbf{1}$, $\partial_{\mathbf{T}}\,\mathrm{tr}\,[\mathbf{T}^2]/2 = \mathbf{T}^T$ and $\partial_{\mathbf{T}}\,\mathrm{tr}\,[\mathbf{T}^3]/3 = \mathbf{T}^{T^2}$, we evaluate the constitutive functions in Table 1 and get the expressions

$$\mathbf{S} = \sum_{i=1}^{3} 2\hat{\psi}_{,i}\mathbf{C}^{-1}\mathbf{E}^i \quad \text{and} \quad \boldsymbol{\tau} = \sum_{i=1}^{3} 2\hat{\psi}_{,i}\mathbf{g}^{-1}\mathbf{e}^i \tag{80}$$

for the stresses,

$$\mathbf{S}^p := \tfrac{1}{2}\mathbf{G}^{p-1}\mathbf{C}\mathbf{S} \quad \text{and} \quad \boldsymbol{\tau}^p := \tfrac{1}{2}\mathbf{c}^{p-1}\mathbf{g}\boldsymbol{\tau} \tag{81}$$

for the plastic forces and

$$B := -\hat{\psi}_{,A} \tag{82}$$

for the conjugate internal variable. Here we have used the notation $\hat{\psi}_{,i} := \partial_{I_i}\hat{\psi}$ and $\hat{\psi}_{,A} := \partial_A\hat{\psi}$ for the derivatives of the free energy by the invariants and the internal variable. Note that the introduction of the variables (79) for the description of the elastic response induce the relationships (81) between the plastic forces and the stresses, see also $(73)_3$. This is of great importance with regard to the identification of the yield criterion function, which has been formulated within the canonical framework summarized in Table 1 in terms of the plastic forces. Here we assume in analogy to (77) a dependence on the invariants $\{S_i\}_{i=1,3}$ of the plastic forces and the generic conjugate internal variable $B$ defined in (82). Thus,

$$\boxed{\phi = \hat{\phi}(S_1, S_2, S_3, B)} \tag{83}$$

is the particular form of (22) which is now under consideration. The invariants $\{S_i\}_{i=1,3}$ of the plastic force are evaluated with respect to the plastic metric. Therefore, we introduce the ground-invariants

$$\left.\begin{aligned} S_1 &:= \mathrm{tr}\,[\boldsymbol{\Sigma}] \\ S_2 &:= \mathrm{tr}\,[\boldsymbol{\Sigma}^2]/2 \\ S_3 &:= \mathrm{tr}\,[\boldsymbol{\Sigma}^3]/3 \end{aligned}\right\} \quad \text{or} \quad \left.\begin{aligned} S_1 &:= \mathrm{tr}\,[\boldsymbol{\sigma}] \\ S_2 &:= \mathrm{tr}\,[\boldsymbol{\sigma}^2]/2 \\ S_3 &:= \mathrm{tr}\,[\boldsymbol{\sigma}^3]/3 \end{aligned}\right\} \tag{84}$$

in terms of the Lagrangian and Eulerian mixed-variant driving stress tensors

$$\boldsymbol{\Sigma} := 2\mathbf{G}^p\mathbf{S}^p = \mathbf{CS} \quad \text{and} \quad \boldsymbol{\sigma} := 2\mathbf{c}^p\boldsymbol{\tau}^p = \mathbf{g}\boldsymbol{\tau} \tag{85}$$

which are due to (81) related to the mixed-variant stresses $\mathbf{CS}$ and $\mathbf{g}\boldsymbol{\tau}$ in the Lagrangian and Eulerian geometric setting, respectively, as already pointed out in $(75)_3$. Taking into account the derivatives of the ground invariants discussed above, we exploit the constitutive functions for the evolution equations in Table 1 in a straightforward manner. This results in the representation

$$\partial_t \mathbf{G}^p = \lambda \sum_{i=1}^{3} 2\hat{\phi}_{,i}\mathbf{G}^p(\boldsymbol{\Sigma}^T)^{i-1} \quad \text{and} \quad \mathscr{L}_v \mathbf{c}^p = \lambda \sum_{i=1}^{3} 2\hat{\phi}_{,i}\mathbf{c}^p(\boldsymbol{\sigma}^T)^{i-1} \tag{86}$$

for the plastic flow rules and

$$\partial_t A = \lambda\hat{\phi}_{,B} \tag{87}$$

for the scalar internal variable with the notation $\hat{\phi}_{,i} := \partial_{S_i}\hat{\phi}$ and $\hat{\phi}_{,B} := \partial_B\hat{\phi}$. The plastic parameter $\lambda$ is determined by the loading–unloading conditions in Table 1. This rounds off the particular set of constitutive equations for isotropic elastoplastic response formulated in terms of invariants.

### 5.2. *Isotropic elastoplasticity in terms of eigenvalues*

We specify the set of constitutive elastoplastic equations summarized in Table 1 to isotropic response formulated in terms of the elastic principal stretches and principal stresses. This treatment results in a representation of constitutive response functions in terms of the right and left eigenvectors of the mixed-variant elastic strain tensors introduced in (79). Thus, we consider the eigenvalue problems

$$\left.\begin{aligned} \mathbf{N}^i\mathbf{E} &= \lambda_i^2\mathbf{N}^i \\ \mathbf{E}\mathbf{N}_i &= \lambda_i^2\mathbf{N}_i \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} \mathbf{n}^i\mathbf{e} &= \lambda_i^2\mathbf{n}^i \\ \mathbf{e}\mathbf{n}_i &= \lambda_i^2\mathbf{n}_i \end{aligned}\right\}. \tag{88}$$

Here $\{\lambda_i\}_{i=1\ldots3}$ are the elastic principal stretches. $\{\mathbf{N}^i\}_{i=1\ldots3} \in T_X\mathscr{B}$ and $\{\mathbf{N}_i\}_{i=1\ldots3} \in T_X^*\mathscr{B}$ are the dual sets of Lagrangian right and left eigenvectors. $\{\mathbf{n}^i\}_{i=1\ldots3} \in T_x\mathscr{S}$ and $\{\mathbf{n}_i\}_{i=1\ldots3} \in T_x^*\mathscr{S}$ are their Eulerian counterparts. The Eulerian eigenvectors are connected with their Lagrangian counterparts via the deformation map (1) and the tangent and normal maps (2)

$$\mathbf{n}^i = \mathbf{F}\mathbf{N}^i \circ \varphi^{-1} \quad \text{and} \quad \mathbf{n}_i = \mathbf{F}^{-T}\mathbf{N}_i \circ \varphi^{-1}. \tag{89}$$

Note that the dual Lagrangian and Eulerian eigenvectors are related via

$$\mathbf{N}^i \cdot \mathbf{N}_j = \delta^i{}_j \quad \text{and} \quad \mathbf{n}^i \cdot \mathbf{n}_j \delta^i{}_j \tag{90}$$

where $\delta^i{}_j$ denotes the Kronecker symbol. We assume in addition a normalization of the eigenvectors with respect to the plastic metric, which plays the role of a reference metric for the elastic response, i.e.

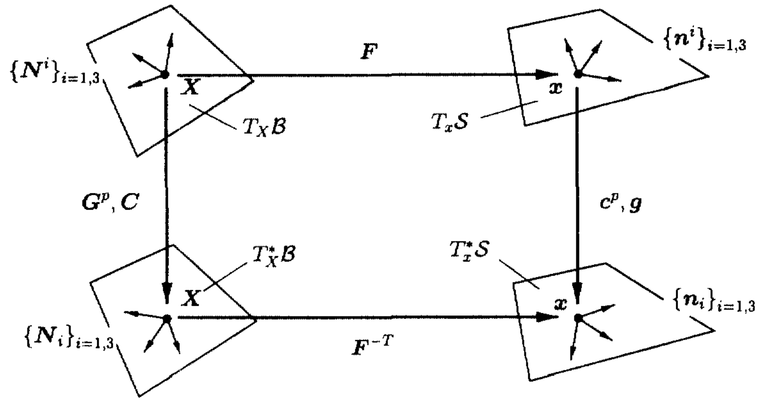



Fig. 6. Eigenvectors associated with local elastic deformation. $\{\mathbf{N}^i\}_{i=1,3}$ and $\{\mathbf{N}_i\}_{i=1,3}$ are the dual Lagrangian eigenvectors with $\mathbf{N}^i \cdot \mathbf{N}_j = \delta^i_j$, $\mathbf{N}_i = \mathbf{G}^p\mathbf{N}^i$, $\lambda_i^2\mathbf{N}_i = \mathbf{C}\mathbf{N}^i$. $\{\mathbf{n}^i\}_{i=1,3}$ and $\{\mathbf{n}_i\}_{i=1,3}$ are the dual Eulerian eigenvectors with $\mathbf{n}^i \cdot \mathbf{n}_j = \delta^i_j$, $\mathbf{n}_i = \mathbf{c}\mathbf{n}^i$ and $\lambda_i^2\mathbf{n}_i = \mathbf{g}\mathbf{n}^i$. Lagrangian and Eulerian eigenvectors are connected according to $\mathbf{n}^i = \mathbf{F}\mathbf{N}^i \circ \varphi^{-1}$ and $\mathbf{n}_i = \mathbf{F}^{-T}\mathbf{N}_i \circ \varphi^{-1}$.

$$\mathbf{N}^i \cdot \mathbf{G}^p\mathbf{N}^i = 1 \quad \text{and} \quad \mathbf{n}^i \cdot \mathbf{c}^p\mathbf{n}^i = 1. \tag{91}$$

(90) and (91) induce the relationships $\mathbf{N}_i = \mathbf{G}^p\mathbf{N}^i$ and $\mathbf{n}_i = \mathbf{c}^p\mathbf{n}^i$. Within this context, we view the reference metrics as maps $\mathbf{G}^p : T_X\mathscr{B} \to T^*_X\mathscr{B}$ and $\mathbf{c}^p : T_x\mathscr{S} \to T^*_x\mathscr{S}$, respectively, as visualized in Fig. 6. The normalization (91) with respect to the reference metric generates the alternative normalization

$$\mathbf{N}^i \cdot \mathbf{C}\mathbf{N}^i = \lambda_i^2 \quad \text{and} \quad \mathbf{n}^i \cdot \mathbf{g}\mathbf{n}^i = \lambda_i^2, \tag{92}$$

inducing the relationships $\lambda_i^2\mathbf{N}_i = \mathbf{C}\mathbf{N}^i$ and $\lambda_i^2\mathbf{n}_i = \mathbf{g}\mathbf{n}^i$. Thus, we also can view the current metric as maps $\mathbf{C} : T_X\mathscr{B} \to T^*_X\mathscr{B}$ and $\mathbf{g} : T_x\mathscr{S} \to T^*_x\mathscr{S}$ as illustrated in Fig. 6. The eigenvalue problems (88) yield the spectral representations

$$\mathbf{E} = \sum_{i=1}^{3} \lambda_i^2\mathbf{N}_i \otimes \mathbf{N}^i \quad \text{and} \quad \mathbf{e} = \sum_{i=1}^{3} \lambda_i^2\mathbf{n}_i \otimes \mathbf{n}^i \tag{93}$$

of the mixed-variant elastic strain tensors (79). Using the normalizations (91) and (92), we derive from (93) the spectral forms

$$\left.\begin{aligned} \mathbf{C} &= \sum_{i=1}^{3} \lambda_i^2\mathbf{N}_i \otimes \mathbf{N}_i \\ \mathbf{G}^p &= \sum_{i=1}^{3} \mathbf{N}_i \otimes \mathbf{N}_i \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} \mathbf{g} &= \sum_{i=1}^{3} \lambda_i^2\mathbf{n}_i \otimes \mathbf{n}_i \\ \mathbf{c}^p &= \sum_{i=1}^{3} \mathbf{n}_i \otimes \mathbf{n}_i \end{aligned}\right\}. \tag{94}$$

We assume now a particular dependence of the isotropic free energy

$$\boxed{\psi = \hat{\psi}(\varepsilon_1, \varepsilon_2, \varepsilon_3, A)} \tag{95}$$

on the logarithmic elastic principal strains $\{\varepsilon_i\}_{i=1\ldots3}$ defined by

$$\varepsilon_i := \tfrac{1}{2}\ln[\lambda_i^2] \tag{96}$$

and the generic scalar internal variable $A$. Taking into account the derivatives $\partial_{\mathbf{E}}\lambda_i^2 = \mathbf{N}^i \otimes \mathbf{N}_i$ and $\partial_{\mathbf{e}}\lambda_i^2 = \mathbf{n}^i \otimes \mathbf{n}_i$ of the principal strains by the mixed-variant elastic strain tensor, we exploit the constitutive expressions in Table 1 in a straightforward manner. This results in the spectral representation

$$\mathbf{S} = \sum_{i=1}^{3} \tau_i/\lambda_i^2 \mathbf{N}^i \otimes \mathbf{N}^i \quad \text{and} \quad \boldsymbol{\tau} = \sum_{i=1}^{3} \tau_i/\lambda_i^2 \mathbf{n}^i \otimes \mathbf{n}^i \tag{97}$$

of the Lagrangian and Eulerian stresses in terms of the principal stresses

$$\tau_i := \hat{\psi}_{,i} \tag{98}$$

in the eigenvalue space.

Furthermore, we get again the result

$$\mathbf{S}^p := \tfrac{1}{2}\mathbf{G}^{p-1}\mathbf{CS} \quad \text{and} \quad \boldsymbol{\tau}^p := \tfrac{1}{2}\mathbf{c}^{p-1}\mathbf{g}\boldsymbol{\tau} \tag{99}$$

for the plastic forces and

$$B := -\hat{\psi}_{,A} \tag{100}$$

for the conjugate internal variable. Here we have used the notation $\hat{\psi}_{,i} := \partial_{\varepsilon_i}\hat{\psi}$ and $\hat{\psi}_{,A} := \partial_A\hat{\psi}$. Insertion of (97) into (99) yields with (94) the spectral form of the plastic forces. We assume again an isotropic dependence of the yield criterion function on the mixed-variant driving stress tensors

$$\boldsymbol{\Sigma} := 2\mathbf{G}^p\mathbf{S}^p = \mathbf{CS} \quad \text{and} \quad \boldsymbol{\sigma} := 2\mathbf{c}^p\boldsymbol{\tau}^p = \mathbf{g}\boldsymbol{\tau} \tag{101}$$

which are due to (99) related to the mixed-variant stresses **CS** and $\mathbf{g}\boldsymbol{\tau}$ in the Lagrangian and Eulerian geometric setting, respectively. These tensors have the spectral form

$$\boldsymbol{\Sigma} = \sum_{i=1}^{3} \tau_i \mathbf{N}_i \otimes \mathbf{N}^i \quad \text{and} \quad \boldsymbol{\sigma} = \sum_{i=1}^{3} \tau_i \mathbf{n}_i \otimes \mathbf{n}^i \tag{102}$$

which induce the formulation

$$\boxed{\phi = \hat{\phi}(\tau_1, \tau_2, \tau_3, B)} \tag{103}$$

of the yield criterion function in terms of the principal stresses and the conjugate internal variable $B$. Based on this assumption we exploit the constitutive functions for the evolution equations summarized in Table 1. Taking into account $\partial_{\boldsymbol{\Sigma}}\tau_i = \mathbf{N}^i \otimes \mathbf{N}_i$ and $\partial_{\boldsymbol{\sigma}}\tau_i = \mathbf{n}^i \otimes \mathbf{n}_i$ we get the representations

$$\partial_t \mathbf{G}^p = \lambda \sum_{i=1}^{3} 2\hat{\phi}_{,i}\mathbf{N}_i \otimes \mathbf{N}_i \quad \text{and} \quad \mathscr{L}_v \mathbf{c}^p = \lambda \sum_{i=1}^{3} 2\hat{\phi}_{,i}\mathbf{n}_i \otimes \mathbf{n}_i \tag{104}$$

of the flow rule in spectral form and the evolution equation

$$\partial_t A = \lambda \hat{\phi}_{,B} \tag{105}$$

for the scalar internal variable with $\hat{\phi}_{,i} := \partial_{\tau_i}\hat{\phi}$ and $\hat{\phi}_{,B} := \partial_B\hat{\phi}$. The plastic parameter $\lambda$ is determined by the loading–unloading conditions in Table 1. This rounds off the particular set of constitutive equations for isotropic elastoplastic response formulated in terms of eigenvalues. For an additional reading concerning formulations in principal strains we refer to Ogden (1984), Miehe (1994b, 1995b) and references therein.

## 6. APPLICATION TO INITIALLY ANISOTROPIC ELASTOPLASTIC RESPONSE

In this Section we consider specifications of the constitutive equations proposed in Sections 2 and 3 to initially anisotropic elastoplastic response. As already pointed out in Section 2, we describe effects of initial anisotropy by isotropic tensor functions with an extended set of arguments. Thereby, we consider as a typical model problem the anisotropy tensors in (5) and (8) as geometric structural variables. These geometric structural variables are *a priori* given and therefore, not governed by a constitutive evolution equation. This is in line with representations of anisotropic response functions proposed by Spencer (1971), Smith (1971), Betten (1982), Boehler (1987) and others. In the following two subsections we discuss on this basis, possible approaches to the modelling of initially elastic and plastic anisotropic response within both the general framework outlined in Section 2 and the decoupled volumetric–isochoric framework of Section 3.

### 6.1. *General initial anisotropic elastoplastic response*

We discuss first a setting of initial anisotropic elastoplastic response within the general framework outlined in Section 2. Here we turn our attention in succession to the description of elastic anisotropy and plastic anisotropy, respectively, and demonstrate the conceptual approach for the case of transverse initial isotropy.

6.1.1. *General elastic anisotropic response.* A particular class of anisotropic elastic response can be characterized by formulation of the free energy function (13) in terms of the second-order anisotropy variables $\mathbf{A}$ and $\boldsymbol{\alpha}$ introduced in (5). The geometric structural variable $\mathbf{A}$ is considered as *a priori* given. This character of the variables for initial anisotropy can be expressed by writing

$$\partial_t \mathbf{A} = 0 \quad \text{and} \quad \mathscr{L}_v \boldsymbol{\alpha} = 0. \tag{106}$$

$\mathbf{A}(X)$ determines for instance a preferred material orientation with respect to the reference configuration. Then $\boldsymbol{\alpha}(x)$ is its convected form with respect to the Eulerian configuration. The anisotropy variables are invariant with respect to rotations $\mathbf{Q} \in \mathscr{G}_3^e$ of the symmetry group $\mathscr{G}_3^e \subset SO_3$ which characterizes the elastic response of the elastoplastic material under consideration. The invariance is expressed by the condition

$$\mathbf{QAQ}^{-1} = \mathbf{A} \quad \text{and} \quad \mathbf{q}\boldsymbol{\alpha}\mathbf{q}^{-1} = \boldsymbol{\alpha} \quad \text{with} \quad \mathbf{q} := \mathbf{F}^{-T}\mathbf{QF}^{T} \circ \varphi^{-1} \quad \forall \mathbf{Q} \in \mathscr{G}_3^e \tag{107}$$

within the Lagrangian and the Eulerian geometric setting, respectively. Note that the anisotropy is formulated with respect to the reference configuration. Thus, the Eulerian form of the invariance condition (107) is nothing more than the convected form of the Lagrangian invariance condition based on the deformation of the Eulerian tensor $\mathbf{q}$ with the property $\mathbf{q}^{-1} = \mathbf{q}$, which depends on the local deformation.

As a concrete model problem we consider here the case of transversely isotropic response. Then the anisotropy variables are given by the expressions

$$\mathbf{A} := \mathbf{M}^e \otimes \mathbf{M}^e \quad \text{and} \quad \boldsymbol{\alpha} := \mathbf{m}^e \otimes \mathbf{m}^e \quad \text{with} \quad \mathbf{m}^e := \mathbf{F}^{-T}\mathbf{M}^e \circ \varphi^{-1} \tag{108}$$

in terms of the given covariant Lagrangian vector field $\mathbf{M}^e \in T_X^*\mathscr{B}$ at $X \in \mathscr{B}$ with the normalization $\mathbf{M}^e \cdot \mathbf{G}^{-1} \cdot \mathbf{M}^e = 1$. This covariant vector is mapped by (1) and $(2)_2$ to the covariant Eulerian vector field $\mathbf{m}^e \in T_x^*\mathscr{S}$ at $x \in \mathscr{S}$, which has the normalization constraint

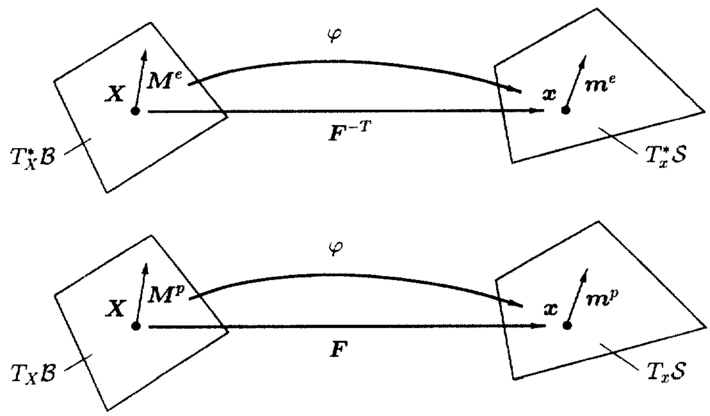



Fig. 7. Anisotropy variables for general transverse anisotropy. The given covariant Lagrangian anisotropy director $\mathbf{M}^e$ with Eulerian counterpart $\mathbf{m}^e = \mathbf{F}^{-T}\mathbf{M}^e \circ \varphi^{-1}$ describes an elastic anisotropy direction. The given contravariant Lagrangian anisotropy director $\mathbf{M}^p$ with Eulerian counterpart $\mathbf{m}^p = \mathbf{F}\mathbf{M}^p \circ \varphi^{-1}$ describes a plastic anisotropy direction.

$\mathbf{m}^e \cdot \mathbf{c}^{-1} \cdot \mathbf{m}^e = 1$. Figure 7 visualizes these geometrical relationships. The symmetry group $\mathscr{G}_3^e$ of the transversely isotropic elastic response is given by the rotations

$$\mathscr{G}_3^e := \{\mathbf{R}_{\|\mathbf{M}^e} = SO_2, \mathbf{R}^\pi{}_{\perp\mathbf{M}^e}\} \tag{109}$$

which leaves (108) invariant. Here $\mathbf{R}_{\|\mathbf{M}^e} = SO_2$ are the arbitrary rotations about an axis with director $\mathbf{M}^e$ and $\mathbf{R}^\pi{}_{\perp\mathbf{M}^e}$ denotes a rotation about a vector perpendicular to $\mathbf{M}^e$ by the angle $\pi$. We discuss in what follows different approaches to anisotropic elastic response within an elastoplastic solid for this simple example of transverse isotropy. More complicated anisotropic response can be described by equipping the second-order tensors $\mathbf{A}$ with more structure than (108), or even by taking into account higher order anisotropy tensors. But the approach to the construction of advanced anisotropic response is conceptually identical to that demonstrated here.

For transversely anisotropic elastoplastic response we assume a dependence of the free energy on the coupled invariants $\{I_i\}_{i=1,5}$ of the elastic strain tensor $(72)_2$ and the anisotropy tensor (108). We restrict our consideration here to a formulation of ideal elastoplasticity. The model can easily be extended to a description of isotropic and kinematic hardening effects by taking into account, in addition, the hardening variables discussed in Sections 5 and 7. Thus, we consider the particular form

$$\psi = \hat{\psi}(I_1, I_2, I_3, I_4, I_5) \tag{110}$$

of (13). The invariants $\{I_i\}_{i=1,5}$ are defined by

$$\left.\begin{aligned} I_1 &:= \operatorname{tr}[\mathbf{EG}] \\ I_2 &:= \operatorname{tr}[(\mathbf{EG})^2]/2 \\ I_3 &:= \operatorname{tr}[(\mathbf{EG})^3]/3 \\ I_4 &:= \operatorname{tr}[\mathbf{EA}] \\ I_5 &:= \operatorname{tr}[\mathbf{EGEA}] \end{aligned}\right\} \quad \text{or} \quad \left.\begin{aligned} I_1 &:= \operatorname{tr}[\mathbf{ec}] \\ I_2 &:= \operatorname{tr}[(\mathbf{ec})^2]/2 \\ I_3 &:= \operatorname{tr}[(\mathbf{ec})^3]/3 \\ I_4 &:= \operatorname{tr}[\mathbf{e}\boldsymbol{\alpha}] \\ I_5 &:= \operatorname{tr}[\mathbf{ece}\boldsymbol{\alpha}] \end{aligned}\right\} \tag{111}$$

in terms of the Lagrangian and Eulerian elastic strain tensors

$$\mathbf{E} := \mathbf{G}^{p-1}\mathbf{C}\mathbf{G}^{p-1} \quad \text{and} \quad \mathbf{e} := \mathbf{c}^{p-1}\mathbf{g}\mathbf{c}^{p-1} \tag{112}$$

which has been introduced in $(72)_2$. Based on the assumption (110) we evaluate the constitutive functions in Table 1 and the get the representations

$$\begin{aligned}\mathbf{S} &= \underbrace{\mathbf{G}^{p-1}\,[\Sigma_{i=1}^{3}\,2\hat{\psi}_{,i}\mathbf{G}(\mathbf{EG})^{i-1}]\mathbf{G}^{p-1}}_{\mathbf{S}_{\text{isotrop}}} \\ &\quad + \underbrace{\mathbf{G}^{p-1}[2\hat{\psi}_{,4}\mathbf{A} + 2\hat{\psi}_{,5}(\mathbf{AEG}+\mathbf{GEA})]\mathbf{G}^{p-1}}_{\mathbf{S}_{\text{anisotrop}}} \\ \boldsymbol{\tau} &= \underbrace{\mathbf{c}^{p-1}\,[\Sigma_{i=1}^{3}\,2\hat{\psi}_{,i}\mathbf{c}(\mathbf{ec})^{i-1}]\mathbf{c}^{p-1}}_{\boldsymbol{\tau}_{\text{isotrop}}} \\ &\quad + \underbrace{\mathbf{c}^{p-1}[2\hat{\psi}_{,4}\boldsymbol{\alpha} + 2\hat{\psi}_{,5}(\boldsymbol{\alpha}\mathbf{ec}+\mathbf{ce}\boldsymbol{\alpha})]\mathbf{c}^{p-1}}_{\boldsymbol{\tau}_{\text{anisotrop}}} \end{aligned} \tag{113}$$

for the stresses and

$$\mathbf{S}^p := \operatorname{sym}[\mathbf{G}^{p-1}\mathbf{CS}] \quad \text{and} \quad \boldsymbol{\tau}^p := \operatorname{sym}[\mathbf{c}^{p-1}\mathbf{g}\boldsymbol{\tau}] \tag{114}$$

for the plastic forces. Recall that the introduction of the variables (112) for the description of the elastic response induces the relationships (114) between the plastic forces and the stresses, see also $(73)_2$ in Section 4. Note furthermore, that the stresses can always be split up additively into isotropic parts and anisotropic parts where the latter contain the anisotropy variables. This property is very helpful with regard to extensions of isotropic elastic models to anisotropic elastic response. Due to the formulation of the coupled invariants $I_4$ and $I_5$ in (111) in terms of the elastic strain tensors (112) the anisotropy is defined with respect to a plastic intermediate configuration, see Fig. 5 and the discussion in Section 4. This assumption makes sense in the case of metal plasticity, where the anisotropy properties are often assumed to be independent of the local plastic deformation. Alternatively, one could define the coupled invariants in (111) in terms of the current metric by setting for instance $I_4 := \operatorname{tr}[\mathbf{G}^{-1}\mathbf{CG}^{-1}\mathbf{A}]$ and $I_5 := \operatorname{tr}[\mathbf{G}^{-1}\mathbf{CG}^{-1}\mathbf{CG}^{-1}\mathbf{A}]$ in the Lagrangian form. Then the elastic initial anisotropy is defined with respect to the reference configuration which yields the alternative representations $\mathbf{S}_{\text{anisotrop}} := \mathbf{G}^{-1}[2\hat{\psi}_{,4}\mathbf{A} + 2\hat{\psi}_{,5}(\mathbf{AG}^{-1}\mathbf{C} + \mathbf{CG}^{-1}\mathbf{A})\mathbf{G}^{-1}$ and $\mathbf{S}^p := \operatorname{sym}[\mathbf{G}^{p-1}\mathbf{CS}_{\text{isotrop}}]$ in (113) and (114), respectively. This could describe a material consisting of elastic fibers within an isotropic elastoplastic matrix.

6.1.2. *General plastic anisotropic response.* In analogy to the elastic anisotropy discussed above, we consider a particular class of anisotropic plastic response by formulating the flow criterion function (22) in terms of the second-order anisotropy variables $\mathbf{B}$ and $\boldsymbol{\beta}$ introduced in (8). The geometric structural variable $\mathbf{B}$ is considered as *a priori* given and describes for instance an anisotropy director with respect to the reference configuration. Thus, we have

$$\partial_t \mathbf{B} = 0 \quad \text{and} \quad \mathscr{L}_v \boldsymbol{\beta} = 0. \tag{115}$$

The anisotropy variable $\mathbf{B}(X)$ and its convected form $\boldsymbol{\beta}(x)$ are invariant under rotations $\mathbf{Q} \in \mathscr{G}_3^p$ of the symmetry group $\mathscr{G}_3^p \subset SO_3$ which characterizes the plastic response of the material under consideration. The invariance is expressed by the condition

$$\mathbf{QBQ}^{-1} = \mathbf{B} \quad \text{and} \quad \mathbf{q}\boldsymbol{\beta}\mathbf{q}^{-1} = \boldsymbol{\beta} \quad \text{with} \quad \mathbf{q} := \mathbf{F}^{-T}\mathbf{QF}^{T} \circ \varphi^{-1} \quad \forall \mathbf{Q} \in \mathscr{G}_3^p \tag{116}$$

within the Lagrangian and Eulerian setting, respectively. As a model problem we consider here again the case of transversely isotropic response where the anisotropy variables are given by the expressions

$$\mathbf{B} := \mathbf{M}^p \otimes \mathbf{M}^p \quad \text{and} \quad \boldsymbol{\beta} := \mathbf{m}^p \otimes \mathbf{m}^p \quad \text{with} \quad \mathbf{m}^p := \mathbf{FM}^p \circ \varphi^{-1} \tag{117}$$

in terms of the given contravariant Lagrangian unit vector $\mathbf{M}^p \in T_X\mathscr{B}$ at $X \in \mathscr{B}$ with the normalization $\mathbf{M}^p \cdot \mathbf{G} \cdot \mathbf{M}^p = 1$. This contravariant vector is mapped by (1) and $(2)_1$ to the

Eulerian vector $\mathbf{m}^p \in T_x\mathcal{S}$ at $x \in \mathcal{S}$ which has the normalization constraint $\mathbf{m}^p \cdot \mathbf{c} \cdot \mathbf{m}^p = 1$, see also Fig. 7. The symmetry group $\mathcal{G}_3^p$ of the transversely isotropic plastic response is given by the rotations

$$\mathcal{G}_3^p := \{\mathbf{R}_{\|\mathbf{M}^p} = SO_2, \mathbf{R}^{\pi}{}_{\perp\mathbf{M}^p}\} \tag{118}$$

where $\mathbf{R}_{\|\mathbf{M}^p} = SO_2$ are arbitrary rotations about an axis with director $\mathbf{M}^p$ and $\mathbf{R}^{\pi}{}_{\perp\mathbf{M}^p}$ denotes a rotation about a vector perpendicular to $\mathbf{M}^p$ by the angle $\pi$. Clearly, in many cases it will make sense to choose for the plastic response the identical symmetry group as for the elastic response by setting $\mathcal{G}_3^p \equiv \mathcal{G}_3^e$.

For transversely anisotropic elastoplastic response we assume a dependence of the flow criterion function on the coupled invariants $\{S_i\}_{i=1,5}$ of driving stress $(75)_2$ and the anisotropy tensor (117). By restricting ourselves to the case of ideal elastoplasticity, we then consider the particular form

$$\boxed{\phi = \hat{\phi}(S_1, S_2, S_3, S_4, S_5)} \tag{119}$$

of (22). The invariants $\{S_i\}_{i=1,5}$ are defined by

$$\left.\begin{array}{l} S_1 := \operatorname{tr}[\mathbf{\Sigma}\mathbf{G}^{p-1}] \\ S_2 := \operatorname{tr}[(\mathbf{\Sigma}\mathbf{G}^{p-1})^2]/2 \\ S_3 := \operatorname{tr}[(\mathbf{\Sigma}\mathbf{G}^{p-1})^3]/3 \\ S_4 := \operatorname{tr}[\mathbf{\Sigma}\mathbf{B}] \\ S_5 := \operatorname{tr}[\mathbf{\Sigma}\mathbf{G}^{p-1}\mathbf{\Sigma}\mathbf{B}] \end{array}\right\} \quad \text{or} \quad \left.\begin{array}{l} S_1 := \operatorname{tr}[\boldsymbol{\sigma}\mathbf{c}^{p-1}] \\ S_2 := \operatorname{tr}[(\boldsymbol{\sigma}\mathbf{c}^{p-1})^2]/2 \\ S_3 := \operatorname{tr}[(\boldsymbol{\sigma}\mathbf{c}^{p-1})^3]/3 \\ S_4 := \operatorname{tr}[\boldsymbol{\sigma}\boldsymbol{\beta}] \\ S_5 := \operatorname{tr}[\boldsymbol{\sigma}\mathbf{c}^{p-1}\boldsymbol{\sigma}\boldsymbol{\beta}] \end{array}\right\} \tag{120}$$

in terms of the Lagrangian and Eulerian driving stress tensors

$$\mathbf{\Sigma} := \mathbf{G}^p\mathbf{S}^p\mathbf{G}^p = \operatorname{sym}[\mathbf{C}\mathbf{S}\mathbf{G}^p] \quad \text{and} \quad \boldsymbol{\sigma} := \mathbf{c}^p\boldsymbol{\tau}^p\mathbf{c}^p = \operatorname{sym}[\mathbf{g}\boldsymbol{\tau}\mathbf{c}^p] \tag{121}$$

which have been introduced in $(75)_2$. It is the symmetrization of the mixed-variant stresses $\mathbf{CS}$ and $\mathbf{g}\boldsymbol{\tau}$ with respect to the plastic metric $\mathbf{G}^p$ and $\mathbf{c}^p$, respectively. Based on the assumption (119) we evaluate the constitutive expressions for the evolution equations in Table 1 and get the representations

$$\begin{aligned} \partial_t\mathbf{G}^p &= \lambda\{\underbrace{\mathbf{G}^p[\Sigma_{i=1}^3 2\hat{\phi}_{,i}\mathbf{G}^{p-1}(\mathbf{\Sigma}\mathbf{G}^{p-1})^{i-1}]\mathbf{G}^p}_{\mathbf{N}_{\text{isotrop}}} \\ &\quad + \underbrace{\mathbf{G}^p[2\hat{\phi}_{,4}\mathbf{B} + 2\hat{\phi}_{,5}(\mathbf{B}\mathbf{\Sigma}\mathbf{G}^{p-1} + \mathbf{G}^{p-1}\mathbf{\Sigma}\mathbf{B})]\mathbf{G}^p}_{\mathbf{N}_{\text{anisotrop}}}\} \\ \mathcal{L}_v\mathbf{c}^p &= \lambda\{\underbrace{\mathbf{c}^p[\Sigma_{i=1}^3 2\hat{\phi}_{,i}\mathbf{c}^{p-1}(\boldsymbol{\sigma}\mathbf{c}^{p-1})^{i-1}]\mathbf{c}^p}_{\mathbf{N}_{\text{isotrop}}} \\ &\quad + \underbrace{\mathbf{c}^p[2\hat{\phi}_{,4}\boldsymbol{\beta} + 2\hat{\phi}_{,5}(\boldsymbol{\beta}\boldsymbol{\sigma}\mathbf{c}^{p-1} + \mathbf{c}^{p-1}\boldsymbol{\sigma}\boldsymbol{\beta})]\mathbf{c}^p}_{\mathbf{N}_{\text{anisotrop}}}\} \end{aligned} \tag{122}$$

of the plastic flow rules. The plastic parameter $\lambda$ is determined by the loading–unloading conditions in Table 1. The flow directions can be split up additively into isotropic parts and anisotropic parts where the latter is a function of the anisotropy variables. This is an important observation with regard to the construction of anisotropic flow models. As a consequence of the formulation of the coupled invariants $S_4$ and $S_5$ in (120) in terms of the driving stress tensor (121), the plastic anisotropy is defined with respect to a plastic intermediate configuration. As already mentioned above, this assumption makes sense in the case of metal plasticity where the anisotropy properties are often assumed to be independent of the local plastic deformation. Alternatively, one could define the coupled invariants in (120) directly in terms of the plastic force by setting $S_4 := \operatorname{tr}[\mathbf{G}\mathbf{S}^p\mathbf{G}\mathbf{B}]$ and $S_5 := \operatorname{tr}[\mathbf{G}\mathbf{S}\mathbf{G}\mathbf{S}\mathbf{G}\mathbf{B}]$ in the Lagrangian geometric representation. Then the plastic initial anisotropy is defined with respect to the reference configuration and we have the modifications $\mathbf{N}_{\text{anisotrop}} := \mathbf{G}[2\hat{\phi}_{,4}\mathbf{B} + 2\hat{\phi}_{,5}(\mathbf{B}\mathbf{G}\mathbf{S}^p + \mathbf{S}^p\mathbf{G}\mathbf{B})]\mathbf{G}$ in (122), respectively.

6.2. *Isochoric initial anisotropic elastoplastic response*

We consider now a class of initial anisotropic elastoplastic constitutive model which is restricted to the isochoric part of the deformation by applying the frame outlined in Section 3. Here we proceed in the same way as in the subsection above and treat successively elastic anisotropy and plastic anisotropy, respectively.

6.2.1. *Isochoric elastic anisotropic response.* We formulate the isochoric contribution (62) to the free energy in terms of the second-order anisotropy variables $\tilde{\mathbf{A}}$ and $\boldsymbol{\alpha}$ introduced in (47) and (48). The geometric structural variable $\tilde{\mathbf{A}}$ is considered as *a priori* given which induces the properties

$$\partial_t \tilde{\mathbf{A}} = 0 \quad \text{and} \quad \tilde{\mathscr{L}}_v \boldsymbol{\alpha} = 0. \tag{123}$$

$\tilde{\mathbf{A}}(X)$ determines possibly a preferred material orientation with respect to the volumetric intermediate configuration, see Figs 3 and 4. $\boldsymbol{\alpha}(x)$ is the convected form with respect to the Eulerian configuration. The anisotropy variables are invariant with respect to rotations $Q \in \mathscr{G}_3^e$ of the symmetry group $\mathscr{G}_3^e \subset SO_3$, i.e.

$$\mathbf{Q}\tilde{\mathbf{A}}\mathbf{Q}^{-1} \quad \text{and} \quad \mathbf{q}\boldsymbol{\alpha}\mathbf{q}^{-1} = \boldsymbol{\alpha} \quad \text{with} \quad \tilde{\mathbf{q}} := \tilde{\mathbf{F}}^{-T}\mathbf{Q}\tilde{\mathbf{F}}^T \circ \varphi^{-1} \forall \mathbf{Q} \in \mathscr{G}_3^e \tag{124}$$

within the Lagrangian and the Eulerian geometric setting, respectively. The Eulerian form of the invariance condition (107) is nothing more than the convected form of the invariance condition with respect to the volumetric intermediate configuration, transformed by the nonlinear deformation map (1) and the modified tangent and normal maps defined in (46). This defines the Eulerian tensor $\tilde{\mathbf{q}}$ with the property $\tilde{\mathbf{q}}^{-1} = \tilde{\mathbf{q}}$, which depends on the local isochoric part of the deformation.

As a model problem we consider again the case of transversely isotropic response by expressing

$$\tilde{\mathbf{A}} := \tilde{\mathbf{M}}^e \otimes \tilde{\mathbf{M}}^e \quad \text{and} \quad \boldsymbol{\alpha} := \mathbf{m}^e \otimes \mathbf{m}^e \quad \text{with} \quad \mathbf{m}^e := \tilde{\mathbf{F}}^{-T}\tilde{\mathbf{M}}^e \circ \varphi^{-1} \tag{125}$$

in terms of the given covariant Lagrangian vector field $\tilde{\mathbf{M}}^e \in \tilde{T}_X^*\mathscr{B}$ at $X \in \mathscr{B}$ of the volumetric intermediate configuration with the normalization $\tilde{\mathbf{M}}^e \cdot \mathbf{G}^{-1} \cdot \tilde{\mathbf{M}}^e = 1$. This covariant vector is mapped by (1) and $(46)_2$ to the covariant Eulerian vector field $\mathbf{m}^e \in T_x^*\mathscr{S}$ at $x \in \mathscr{S}$, which has the normalization constraint $\mathbf{m}^e \cdot \mathbf{c}^{-1} \cdot \mathbf{m}^e = 1$. See Fig. 8 for a visualization of these

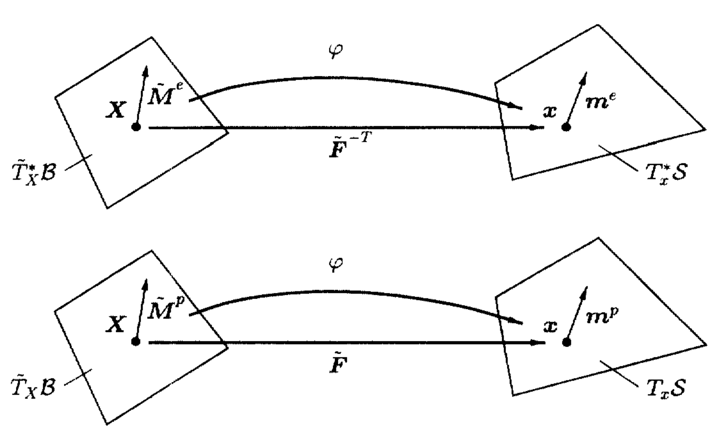



Fig. 8. Anisotropy variables for isochoric transverse anisotropy. The given covariant Lagrangian anisotropy director $\tilde{\mathbf{M}}^e$ with Eulerian counterpart $\mathbf{m}^e = \tilde{\mathbf{F}}^{-T}\tilde{\mathbf{M}}^e \circ \varphi^{-1}$ describes an elastic anisotropy direction. The given contravariant Lagrangian anisotropy director $\tilde{\mathbf{M}}^p$ with Eulerian counterpart $\mathbf{m}^p = \tilde{\mathbf{F}}\tilde{\mathbf{M}}^p \circ \varphi^{-1}$ describes a plastic anisotropy direction.

geometrical relationships. The symmetry group $\mathscr{G}_3^e$ of the transversely isochoric isotropic elastic response is given by the rotations

$$\mathscr{G}_3^e := \{\mathbf{R}_{|\tilde{\mathbf{M}}^e} = SO_2, \mathbf{R}^{\pi}{}_{\perp\tilde{\mathbf{M}}^e}\} \tag{126}$$

which leaves (125) invariant.

For transversely anisotropic elastoplastic response we assume a dependence of the free energy on the coupled invariants $\{\tilde{I}_i\}_{i=1,4}$ of the elastic strain tensor $(72)_2$ and the anisotropy tensor (125). We restrict our consideration again to a formulation of ideal elastoplasticity and consider the form

$$\boxed{\psi_{\text{iso}} = \hat{\psi}_{\text{iso}}(\tilde{I}_1, \tilde{I}_2, \tilde{I}_3, \tilde{I}_4)} \tag{127}$$

of (62). The invariants $\{\tilde{I}_i\}_{i=4,5}$ are defined by

$$\left.\begin{aligned} \tilde{I}_1 &:= \operatorname{tr}[\tilde{\mathbf{E}}\mathbf{G}] \\ \tilde{I}_2 &:= \operatorname{tr}[(\tilde{\mathbf{E}}\mathbf{G})^2]/2 \\ \tilde{I}_3 &:= \operatorname{tr}[\tilde{\mathbf{E}}\tilde{\mathbf{A}}] \\ \tilde{I}_4 &:= \operatorname{tr}[\tilde{\mathbf{E}}\tilde{\mathbf{G}}\tilde{\mathbf{E}}\tilde{\mathbf{A}}] \end{aligned}\right\} \quad \text{or} \quad \left.\begin{aligned} \tilde{I}_1 &:= \operatorname{tr}[\mathbf{ec}] \\ \tilde{I}_2 &:= \operatorname{tr}[(\mathbf{ec})^2]/2 \\ \tilde{I}_3 &:= \operatorname{tr}[\mathbf{e}\boldsymbol{\alpha}] \\ \tilde{I}_4 &:= \operatorname{tr}[\mathbf{ece}\boldsymbol{\alpha}] \end{aligned}\right\} \tag{128}$$

in terms of the Lagrangian and Eulerian elastic strain tensors

$$\tilde{\mathbf{E}} := \tilde{\mathbf{G}}^{p-1}\tilde{\mathbf{C}}\tilde{\mathbf{G}}^{p-1} \quad \text{and} \quad \mathbf{e} := \mathbf{c}^{p-1}\mathbf{g}\mathbf{c}^{p-1} \tag{129}$$

which has been introduced in $(72)_2$. Based on the assumption (127) we evaluate the constitutive functions in Table 3 and get the representations

$$\begin{aligned} \tilde{\mathbf{S}}_{\text{iso}} &= \underbrace{\operatorname{dev}_{\tilde{\mathbf{C}}}\{\tilde{\mathbf{G}}^{p-1}[\Sigma_{i=1}^2 2\hat{\psi}_{\text{iso},i}\mathbf{G}(\tilde{\mathbf{E}}\mathbf{G})^{i-1}]\tilde{\mathbf{G}}^{p-1}\}}_{\mathbf{S}_{\text{iso,isotrop}}} \\ &\quad + \underbrace{\operatorname{dev}_{\tilde{\mathbf{C}}}\{\tilde{\mathbf{G}}^{p-1}[2\hat{\psi}_{\text{iso},3}\tilde{\mathbf{A}} + 2\hat{\psi}_{\text{iso},4}(\tilde{\mathbf{A}}\tilde{\mathbf{E}}\mathbf{G} + \mathbf{G}\tilde{\mathbf{E}}\tilde{\mathbf{A}})]\tilde{\mathbf{G}}^{p-1}\}}_{\mathbf{S}_{\text{iso,anisotrop}}} \\ \boldsymbol{\tau}_{\text{iso}} &= \underbrace{\operatorname{dev}_{\mathbf{g}}\{\mathbf{c}^{p-1}[\Sigma_{i=1}^2 2\hat{\psi}_{\text{iso},i}\tilde{\mathbf{c}}(\mathbf{e}\tilde{\mathbf{c}})^{i-1}]\mathbf{c}^{p-1}\}}_{\boldsymbol{\tau}_{\text{iso,isotrop}}} \\ &\quad + \underbrace{\operatorname{dev}_{\mathbf{g}}\{\mathbf{c}^{p-1}[2\hat{\psi}_{\text{iso},3}\boldsymbol{\alpha} + 2\hat{\psi}_{\text{iso},4}(\boldsymbol{\alpha}\mathbf{e}\tilde{\mathbf{c}} + \tilde{\mathbf{c}}\mathbf{e}\boldsymbol{\alpha})]\mathbf{c}^{p-1}\}}_{\boldsymbol{\tau}_{\text{iso,anisotrop}}} \end{aligned} \tag{130}$$

for the deviatoric stresses and

$$\tilde{\mathbf{S}}^p := \operatorname{sym}[\tilde{\mathbf{G}}^{p-1}\tilde{\mathbf{C}}\tilde{\mathbf{S}}_{\text{iso}}] \quad \text{and} \quad \boldsymbol{\tau}^p := \operatorname{sym}[\mathbf{c}^{p-1}\mathbf{g}\boldsymbol{\tau}_{\text{iso}}] \tag{131}$$

for the deviatoric plastic forces. Recall that (131) transforms the deviators with respect to the current metric to deviators with respect to the plastic metric. We observe again that the stresses can be split up additively into isotropic and anisotropic parts. The anisotropy is formulated with respect to a plastic intermediate configuration as discussed in Section 4. A conceptual approach to a formulation of the anisotropy with respect to the reference configuration has been discussed at the end of Section 6.1.1.

6.2.2. *Isochoric plastic anisotropic response*. Isochoric plastic response can be modelled by formulating the flow criterion function (67) in terms of the second-order anisotropy variables $\tilde{\mathbf{B}}$ and $\boldsymbol{\beta}$ introduced in (47) and (48). The geometric structural variable $\tilde{\mathbf{B}}$ is considered as *a priori* given and describes for instance an anisotropy director with respect to the volumetric intermediate configuration, see Figs 4 and 5. Thus, we have

$$\partial_t \tilde{\mathbf{B}} = 0 \quad \text{and} \quad \mathscr{L}_v \boldsymbol{\beta} = 0. \tag{132}$$

The variables $\tilde{\mathbf{B}}(X)$ and $\boldsymbol{\beta}(x)$ are invariant under rotations $\mathbf{Q} \in \mathscr{G}_3^p$ of the symmetry group $\mathscr{G}_3^p \subset SO_3$, i.e.

$$\mathbf{Q}\tilde{\mathbf{B}}\mathbf{Q}^{-1} = \mathbf{B} \quad \text{and} \quad \tilde{\mathbf{q}}\boldsymbol{\beta}\tilde{\mathbf{q}}^{-1} = \boldsymbol{\beta} \quad \text{with} \quad \tilde{\mathbf{q}} := \tilde{\mathbf{F}}^{-T}\mathbf{Q}\tilde{\mathbf{F}}^{T} \circ \varphi^{-1} \quad \forall \mathbf{Q} \in \mathscr{G}_3^p \tag{133}$$

within the Lagrangian and Eulerian setting, respectively. For transverse isotropy we identify

$$\tilde{\mathbf{B}} := \tilde{\mathbf{M}}^p \otimes \tilde{\mathbf{M}}^p \quad \text{and} \quad \boldsymbol{\beta} := \mathbf{m}^p \otimes \mathbf{m}^p \quad \text{with} \quad \mathbf{m}^p := \tilde{\mathbf{F}}\tilde{\mathbf{M}}^p \circ \varphi^{-1} \tag{134}$$

in terms of the given contravariant vector $\tilde{\mathbf{M}}^p \in \tilde{T}_X\mathscr{B}$ at $X \in \mathscr{B}$ of the volumetric intermediate configuration with the normalization $\tilde{\mathbf{M}}^p \cdot \tilde{\mathbf{G}} \cdot \mathbf{M}^p = 1$. This contravariant vector is mapped by (1) and $(46)_2$ to the Eulerian vector $\mathbf{m}^p \in T_x\mathscr{S}$ at $x \in \mathscr{S}$ which has the normalization constraint $\mathbf{m}^p \cdot \mathbf{c} \cdot \mathbf{m}^p = 1$, see also Fig. 8. The symmetry group $\mathscr{G}_3^p$ of the transversely isotropic plastic response is given by the rotation

$$\mathscr{G}_3^p := \{\mathbf{R}_{\|\tilde{\mathbf{M}}^p} = SO_2, \mathbf{R}^{\pi}{}_{\perp\tilde{\mathbf{M}}^p}\} \tag{135}$$

which leaves (134) invariant. In many cases it will make sense to choose the identical symmetry group for the plastic response as for the elastic response by setting $\mathscr{G}_3^p \equiv \mathscr{G}_3^e$.

For transversely anisotropic elastoplastic response we assume a dependence of the flow criterion function on the coupled invariants $\{\tilde{S}_i\}_{i=1,4}$ of driving stress $(75)_2$ and the anisotropy tensor (134). By restricting it to the case of ideal elastoplasticity, we then consider the particular form

$$\boxed{\phi = \hat{\phi}(\tilde{S}_1, \tilde{S}_2, \tilde{S}_3, \tilde{S}_4)} \tag{136}$$

of (67). The invariants $\{\tilde{S}_i\}_{i=1,4}$ are defined by

$$\left.\begin{aligned} \tilde{S}_1 &:= \operatorname{tr}[(\tilde{\boldsymbol{\Sigma}}\tilde{\mathbf{G}}^{p-1})^2]/2 \\ \tilde{S}_2 &:= \operatorname{tr}[(\tilde{\boldsymbol{\Sigma}}\tilde{\mathbf{G}}^{p-1})^3]/3 \\ \tilde{S}_3 &:= \operatorname{tr}[\tilde{\boldsymbol{\Sigma}}\tilde{\mathbf{B}}] \\ \tilde{S}_4 &:= \operatorname{tr}[\tilde{\boldsymbol{\Sigma}}\tilde{\mathbf{G}}^{p-1}\tilde{\boldsymbol{\Sigma}}\tilde{\mathbf{B}}] \end{aligned}\right\} \quad \text{or} \quad \left.\begin{aligned} \tilde{S}_1 &:= \operatorname{tr}[(\boldsymbol{\sigma}\mathbf{c}^{p-1})^2]/2 \\ \tilde{S}_2 &:= \operatorname{tr}[(\boldsymbol{\sigma}\mathbf{c}^{p-1})^3]/3 \\ \tilde{S}_3 &:= \operatorname{tr}[\boldsymbol{\sigma}\boldsymbol{\beta}] \\ \tilde{S}_4 &:= \operatorname{tr}[\boldsymbol{\sigma}\mathbf{c}^{p-1}\boldsymbol{\sigma}\boldsymbol{\beta}] \end{aligned}\right\} \tag{137}$$

in terms of the Lagrangian and Eulerian deviatoric driving stress tensors

$$\left.\begin{aligned} \tilde{\boldsymbol{\Sigma}} &:= \operatorname{dev}_{\tilde{\mathbf{G}}^p}\{\tilde{\mathbf{G}}^p\tilde{\mathbf{S}}^p\tilde{\mathbf{G}}^p\} = \operatorname{sym}[\tilde{\mathbf{C}}\tilde{\mathbf{S}}_{\text{iso}}\tilde{\mathbf{G}}^p] \\ \boldsymbol{\sigma} &:= \operatorname{dev}_{\mathbf{c}^p}\{\mathbf{c}^p\boldsymbol{\tau}^p\mathbf{c}^p\} = \operatorname{sym}[\mathbf{g}\boldsymbol{\tau}_{\text{iso}}\mathbf{c}^p] \end{aligned}\right\}. \tag{138}$$

It is the symmetrization of the mixed-variant stresses $\mathbf{CS}_{\text{iso}}$ and $\mathbf{g}\boldsymbol{\tau}_{\text{iso}}$ with respect to the plastic metric $\mathbf{G}^p$ and $\mathbf{c}^p$, respectively. Based on the assumption (136) we evaluate the constitutive expressions for the evolution equations in Table 3 and get the representations

$$\begin{aligned} \partial_t \tilde{\mathbf{G}}^p &= \nu\{\underbrace{\operatorname{dev}_{\tilde{\mathbf{G}}^p}\{\tilde{\mathbf{G}}^p[\Sigma_{i=1}^2 2\hat{\phi}_{,i}\tilde{\mathbf{G}}^{p-1}(\tilde{\boldsymbol{\Sigma}}\tilde{\mathbf{G}}^{p-1})^{i-1}]\tilde{\mathbf{G}}^p\}}_{\mathbf{N}_{\text{iso,isotrop}}} \\ &\quad + \underbrace{\operatorname{dev}_{\tilde{\mathbf{G}}^p}\{\mathbf{G}^p[2\hat{\phi}_{,3}\tilde{\mathbf{B}} + 2\hat{\phi}_{,4}(\tilde{\mathbf{B}}\tilde{\boldsymbol{\Sigma}}\tilde{\mathbf{G}}^{p-1} + \tilde{\mathbf{G}}^{p-1}\tilde{\boldsymbol{\Sigma}}\tilde{\mathbf{B}})]\tilde{\mathbf{G}}^p\}}_{\mathbf{N}_{\text{iso,anisotrop}}}\} \\ \mathscr{L}_v \mathbf{c}^p &= \nu\{\underbrace{\operatorname{dev}_{\mathbf{c}^p}\{\mathbf{c}^p[\Sigma_{i=1}^2 2\hat{\phi}_{,i}\mathbf{c}^{p-1}(\boldsymbol{\sigma}\mathbf{c}^{p-1})^{i-1}]\mathbf{c}^p\}}_{\mathbf{n}_{\text{iso,isotrop}}} \\ &\quad + \underbrace{\operatorname{dev}_{\mathbf{c}^p}\{\mathbf{c}^p[2\hat{\phi}_{,3}\boldsymbol{\beta} + 2\hat{\phi}_{,4}(\boldsymbol{\beta}\boldsymbol{\sigma}\mathbf{c}^{p-1} + \mathbf{c}^{p-1}\boldsymbol{\sigma}\boldsymbol{\beta})]\mathbf{c}^p\}}_{\mathbf{n}_{\text{iso,anisotrop}}}\} \end{aligned} \tag{139}$$

of the plastic flow rules. The plastic parameter $\nu$ is determined by the loading–unloading conditions in Table 3. The flow directions can be split up additively into isotropic parts and anisotropic parts, where the latter is a function of the anisotropy variables. For an alternative formulation of the anisotropy with respect to the reference configuration we refer to the discussion at the beginning of Section 6.1.2. Observe furthermore, that the flow rules (139) preserve exactly the plastic volume. They satisfy the conditions

$$\partial_t \tilde{\mathbf{G}}^{p-1} : \tilde{\mathbf{G}}^p = \frac{\partial_t(\det[\tilde{\mathbf{G}}^p])}{\det[\tilde{\mathbf{G}}^p]} = 0 \quad \text{and} \quad \mathscr{L}_v \mathbf{c}^p : \mathbf{c}^{p-1} = \frac{\partial_t(\det[\mathbf{c}^p])}{\det[\mathbf{c}^p]} = 0, \tag{140}$$

which are often referred to as the plastic incompressibility constraint.

## 7. APPLICATION TO INDUCED ANISOTROPIC ELASTOPLASTIC RESPONSE

This Section is devoted to a short conceptual discussion of the modelling of induced anisotropy effects within the framework of elastoplasticity discussed in Sections 2 and 3. The phenomenon of induced anisotropy is an effect which develops during the inelastic deformation process. Thus, in contrast to the case of initial anisotropy discussed above, the anisotropy variables are now not *a priori* given by develop during the inelastic deformation process, governed by constitutive evolution equations. A typical example of induced anisotropy is the damage accumulation due to microcracks if one takes into account its orientated character. Another example is the kinematic hardening phenomenon in metals, the so-called Bauschinger effect, which is a consequence of the texture development in polycrystalline metallic materials.

The framework for the description of induced anisotropy has already been set up in Sections 2 and 3. A particular class of anisotropic elastoplastic materials can be described by the second-order anisotropy variables introduced in (5) and (8). The canonical form of the constitutive evolution equations for these variables have been presented in Tables 2 and 3 for general elastoplastic response and isochoric elastoplastic response, respectively. We consider here as a model problem a particular form of kinematic hardening based on internal micromechanical free energy storage as suggested by Miehe (1996c). It is a straightforward generalization of the classical Melan–Prager type model of the geometric linear theory. Because this phenomenon is mainly associated with metal plasticity, we restrict here our consideration *a priori* to isochoric response and apply the framework outlined in Section 3.

We assume a dependence of the free energy on the invariants $\{\tilde{I}_i\}_{i=1,4}$ of the elastic strain tensor $(72)_3$ and the anisotropy tensor (5) and consider the particular form

$$\boxed{\psi_{\mathrm{iso}} = \hat{\psi}_{\mathrm{iso}}(\tilde{I}_1, \tilde{I}_2, \tilde{I}_3, \tilde{I}_4)} \tag{141}$$

of (62). The invariants $\{\tilde{I}_i\}_{i=1,4}$ are defined by

$$\left.\begin{aligned} \tilde{I}_1 &:= \operatorname{tr}[\tilde{\mathbf{E}}] \\ \tilde{I}_2 &:= \operatorname{tr}[\tilde{\mathbf{E}}^2]/2 \\ \tilde{I}_3 &:= \operatorname{tr}[\tilde{\mathbf{A}}\mathbf{G}^{-1}] \\ \tilde{I}_4 &:= \operatorname{tr}[(\tilde{\mathbf{A}}\mathbf{G}^{-1})^2]/2 \end{aligned}\right\} \quad \text{or} \quad \left\{\begin{aligned} \tilde{I}_1 &:= \operatorname{tr}[\mathbf{e}] \\ \tilde{I}_2 &:= \operatorname{tr}[\mathbf{e}^2]/2 \\ \tilde{I}_3 &:= \operatorname{tr}[\boldsymbol{\alpha}\mathbf{c}^{-1}] \\ \tilde{I}_4 &:= \operatorname{tr}[(\boldsymbol{\alpha}\mathbf{c}^{-1})^2]/2 \end{aligned}\right. \tag{142}$$

in terms of the Lagrangian and Eulerian mixed-variant elastic strain tensors

$$\tilde{\mathbf{E}} := \tilde{\mathbf{C}}\tilde{\mathbf{G}}^{p-1} \quad \text{and} \quad \mathbf{e} := \mathbf{g}\mathbf{c}^{p-1}, \tag{143}$$

see also $(72)_3$ and (79) and the mixed-variant anisotropy variables $\tilde{\mathbf{A}}\mathbf{G}^{-1}$ and $\boldsymbol{\alpha}\mathbf{c}^{-1}$. Note that the Lagrangian representation is formulated in terms of the variables defined in (47) relative to the volumetric intermediate configuration, see also Figs 3 and 4. Based on the assumption (141) we evaluate the constitutive functions in Table 3 and get the representations

$$\tilde{\mathbf{S}}_{\text{iso}} = \text{dev}_{\tilde{\mathbf{C}}}\left[\sum_{i=1}^{2} 2\hat{\psi}_{\text{iso},i}\tilde{\mathbf{C}}^{-1}\tilde{\mathbf{E}}^{i}\right] \quad \text{and} \quad \boldsymbol{\tau}_{\text{iso}} = \text{dev}_{\mathbf{g}}\left[\sum_{i=1}^{2} 2\hat{\psi}_{\text{iso},i}\mathbf{g}^{-1}\mathbf{e}^{i}\right] \tag{144}$$

for the stresses,

$$\tilde{\mathbf{S}}^{p} := \tfrac{1}{2}\tilde{\mathbf{G}}^{p-1}\tilde{\mathbf{C}}\tilde{\mathbf{S}}_{\text{iso}} \quad \text{and} \quad \boldsymbol{\tau}^{p} := \tfrac{1}{2}\mathbf{c}^{p-1}\mathbf{g}\boldsymbol{\tau}_{\text{iso}} \tag{145}$$

for the plastic forces and

$$\tilde{\mathbf{B}} = -\sum_{i=3}^{4} 2\hat{\psi}_{\text{iso},i}\mathbf{G}^{-1}(\tilde{\mathbf{A}}\mathbf{G}^{-1})^{i-3}] \quad \text{and} \quad \boldsymbol{\beta} = -\sum_{i=3}^{4} 2\hat{\psi}_{\text{iso},i}\mathbf{c}^{-1}(\boldsymbol{\alpha}\mathbf{c}^{-1})^{i-3}] \tag{146}$$

for the conjugate anisotropy variable. The stresses (144) are deviatoric with respect to the current metric and the plastic forces (145) are deviatoric with respect to the plastic metric. As a typical assumption for the description of kinematic hardening we consider the yield criterion function

$$\boxed{\phi = \hat{\phi}(\tilde{S}_1, \tilde{S}_2)} \tag{147}$$

as a particular form of (67). The invariants $\{\tilde{S}_i\}_{i=1,2}$ are defined by

$$\left.\begin{aligned}\tilde{S}_1 &:= \text{tr}\,[\tilde{\boldsymbol{\Sigma}}^2]/2\\ \tilde{S}_2 &:= \text{tr}\,[\tilde{\boldsymbol{\Sigma}}^3]/3\end{aligned}\right\} \quad \text{or} \quad \left.\begin{aligned}\tilde{S}_1 &:= \text{tr}\,[\boldsymbol{\sigma}^2]/2\\ \tilde{S}_2 &:= \text{tr}\,[\boldsymbol{\sigma}^3]/3\end{aligned}\right\} \tag{148}$$

in terms of the Lagrangian and Eulerian driving stress tensors

$$\tilde{\boldsymbol{\Sigma}} := \text{dev}_{\tilde{\mathbf{G}}^p}[2\tilde{\mathbf{G}}^p(\tilde{\mathbf{S}}^p + \tilde{\mathbf{B}})] \quad \text{and} \quad \boldsymbol{\sigma} := \text{dev}_{\mathbf{c}^p}[2\mathbf{c}^p(\boldsymbol{\tau}^p + \boldsymbol{\beta})]. \tag{149}$$

The conjugate anisotropy variables $\tilde{\mathbf{B}}$ and $\boldsymbol{\beta}$ play the role of the (negative) back stresses. Note furthermore, that the driving stress (149) is *a priori* assumed to be deviatoric. Then the evaluation of the constitutive functions for the evolution equations in Table 3 gives the representations

$$\partial_t\tilde{\mathbf{G}}^p = v\,\text{dev}_{\tilde{\mathbf{G}}^p}\left[\sum_{i=1}^{2} 2\hat{\phi}_{,i}\tilde{\mathbf{G}}^p(\tilde{\boldsymbol{\Sigma}}^T)^{i-1}\right] \quad \text{and} \quad \tilde{\mathscr{L}}_v\mathbf{c}^p = v\,\text{dev}_{\mathbf{c}^p}\left[\sum_{i=1}^{2} 2\hat{\phi}_{,i}\right]\mathbf{c}^p(\boldsymbol{\sigma}^T)^{i-1} \tag{150}$$

for the plastic metric and

$$\partial_t\tilde{\mathbf{A}} = \partial_t\tilde{\mathbf{G}}^p \quad \text{and} \quad \tilde{\mathscr{L}}_v\boldsymbol{\alpha} = \tilde{\mathscr{L}}_v\mathbf{c}^p \tag{151}$$

for the anisotropy variable. The plastic parameter $v$ is determined by the loading–unloading conditions in Table 3. Assuming identical initial conditions for the anisotropy variables $\tilde{\mathbf{A}}$, $\boldsymbol{\alpha}$ and the plastic metric $\tilde{\mathbf{G}}^p$, $\mathbf{c}^p$, we get from canonical evolution eqn (151) the identification

$$\tilde{\mathbf{A}} = \tilde{\mathbf{G}}^p \quad \text{and} \quad \boldsymbol{\alpha} = \mathbf{c}^p. \tag{152}$$

Thus, for the proposed canonical model the anisotropy variables $\tilde{\mathbf{A}}$ and $\boldsymbol{\alpha}$ turn out to be identical with the plastic metric. This is consistent with the classical model of kinematic hardening suggested by Melan (1939) for the geometric linear theory. Thus, the negative back stresses $\tilde{\mathbf{B}}$ and $\boldsymbol{\beta}$ are defined in terms of the general hyperelastic constitutive function (146) in terms of the plastic metric. Consider as a model problem the ansatz $\psi_{\text{micro}} = k[\tilde{I}_4 - \tilde{I}_3]/2$ of the free energy for the micro-stress storage associated with kinematic hardening where $k > 0$ is a scalar material parameter. Then we derive from (146) with the identification (152) the constitutive expressions $\tilde{\mathbf{B}} = -k[\mathbf{G}^{-1}\tilde{\mathbf{G}}^p\mathbf{G}^{-1} - \mathbf{G}^{-1}]$ and $\boldsymbol{\beta} = -k[\mathbf{c}^{-1}\mathbf{c}^p\mathbf{c}^{-1} - \mathbf{c}^{-1}]$ for the negative back stresses within the Lagrangian and Eulerian representation, respectively. These equations constitute a straightforward generalization of the classical Melan–Prager type kinematic hardening model to the theoretical frame of large-strain elastoplasticity proposed here.

## 8. CONCLUSION

A constitutive framework of large strain elastoplasticity has been presented for general anisotropic material response. The proposed representation has a strong underlying geometric accent with an orientation towards concepts of irreversible thermodynamics. The essential ingredients of the theory are the introduction of a plastic metric with six independent degrees for description of the history-dependent inelastic material response and the definition of a convex elastic domain in the space of the plastic forces conjugate to the plastic metric. The central constitutive functions for the description of the stored free energy and the yield criterion, see (13), (22), (62) and (67), are formulated as isotropic tensor functions in terms of extended arguments, denoted as anisotropy variables. The latter take into account effects of initial and induced anisotropy which we represent in a coordinate-free format. The representation of the constitutive functions are forced to have the identical structure within the Lagrangian and Eulerian geometric setting, thus characterizing a covariant theory. This has been demonstrated by considering both geometric settings in parallel throughout the whole paper. The set-up of canonical evolution equations, constructed on the basis of a thermodynamic extremum principle, results in the constitutive set summarized in Table 1. This canonical set is characterized by symmetric elastoplastic tangent moduli as proved in (39). A further key aspect is the proposed decomposition of the constitutive set in Table 1 into decoupled volumetric and isochoric parts summarized in Tables 2 and 3, respectively. This has been achieved based on the notion of a volumetric intermediate configuration and the split of the free energy and the local dissipation into decoupled volumetric and isochoric contributions, see (51) and (54), yielding the decomposition (52) of the stresses into decoupled spherical and deviatoric parts. This offers for instance, a geometric consistent restriction of anisotropy properties to the isochoric part of the deformation. The proposed constitutive frame of elastoplasticity has been applied to several model problems. In the context, we considered first, possible elastic strain measures and investigated the induced identification of the plastic forces in terms of the stresses, see (72) and (73). The first application was concerned with isotropic elastoplastic response. Here we proposed two new compact settings in terms of invariants and eigenvalues of a mixed-variant elastic strain measure, see (77), (83), (95) and (103), respectively. A key result within this context has been a spectral representation of the isotropic elastoplastic constitutive equations in terms of dual covariant and contravariant eigenvector triads. Finally, we discussed the conceptual modelling of initial and induced anisotropy within the framework proposed here. Here we considered as a model problem the effect of transversely initial anisotropy for general and purely isochoric response see (110), (119), (127) and (136). We then suggested an extension of the classical Melan-type kinematic hardening model to the nonlinear framework proposed here, see (141) and (147). The algorithmic formulation and the numerical implementation associated with the constitutive framework and the model problems presented here will be discussed in a forthcoming paper.

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